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Dynamic Characteristics of Chaotic Processes Determined from Point Process Analysis

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Abstract—The capability of calculating the highest Lyapunov index during analysis of the so-called point processes [1] is analyzed. Two mathematical models describing the generation of pulses by receptor neurons are considered and the conditions are established for which the dynamic characteristics of chaotic oscillations, determined from the output sequence of pulses, are retained during linear transformations of the neuron input signal. © *2000 MAIK "Nauka/Interperiodica".*

The study of information processing in living organisms is a currently important task of natural sciences. Solving this task encounters a large variety of particular problems, including the question of how are data coded by the nerve cells. Each cell (receptor neuron) is essentially a threshold device transforming a complex input signal $S(t)$ into a sequence of identical pulses (spikes) registered at the output (Fig. 1a). Since the shape of the output pulses is independent of the external factors, all information about the input signal $S(t)$ must be converted into the length of time intervals between output pulses—interspike intervals (ISIs) [2, 3].

In how much detail can the input signal be characterized upon analysis of the output sequence of spikes? In recent years, this problem has drawn the attention of researchers in connection with the problem of dynamic system (DS) reconstruction. In order to apply reconstruction methods to analysis of the point processes (where the information is carried in the form of times of various events), it is necessary to answer the question formulated by Sauer [1]: can the output ISI sequence determine the state of a system if the input signal is deterministic and generated by the DS with small-scale dynamics?

An answer to this question was also originally given by Sauer [1], according to which an ISI set can be considered as points of a new coordinate of state, in using which it is possible to characterize the small-scale dynamics at the input. Then, Sauer [4] proved the embedding theorem for time intervals, thus extending rigorous mathematical formalism developed by Takens [5] to the case of point processes. The possibilities of reconstruction were investigated by numerical methods in [6–9].

By now, various models have been developed to describe the process of spike generation, including the rather popular (and biologically justified) "integrate-and-fire" (IF) and "threshold crossing" (TC) models [8]. Within the framework of the IF model, the $S(t)$ signal is frequently represented by a function of variables of the small-scale dynamic system. A set of times T_i corresponding to the moments of spike generation (Fig. 1a) is determined from the equation

$$\int_{T_i}^{T_{i+1}} S(t)dt = \theta, \quad I_i = T_{i+1} - T_i, \qquad (1)$$

where θ is the threshold level and I_i are the time intervals (IF ISIs). The integral value is reset to zero on reaching the threshold.

The TC model introduces a threshold level Θ determining the equation of a secant plane $S = \Theta$, where $S(t)$ is considered as a DS coordinate. The spike production corresponds to the time instants when the threshold level is crossed by the signal $S(t)$ in one direction (e.g., upward). From the standpoint of the DS theory, the time intervals between spikes (TC ISIs) represent the times of phase trajectory return to the secant plane.

In this work, we attempted to answer the question as to how the threshold level and the ISI sequence structure affect the results of the reconstruction. The conversion of a continuous input signal into a sequence of spikes is a nonlinear transformation. Moreover, this transformation is accompanied by a partial loss of information about the external factor (in particular, about the signal shape in the TC model). Can we still calculate characteristics of the input signal using the ISI sequence and what are the necessary conditions providing for this possibility?

The study was focused on calculating the highest Lyapunov index λ_1, which is apparently the most informative invariant of a complex dynamic process. Below, we will discuss the conditions under which the λ_1 value can be determined from analysis of a point process.

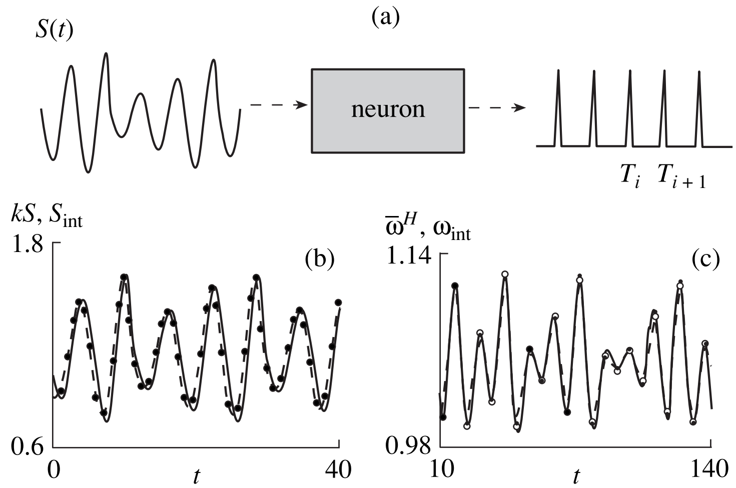

Fig. 1. (a) A schematic diagram illustrating the input signal conversion by a receptor neuron. (b) A linear transformation of the input signal: solid curve, $\frac{1}{\theta}S(t)$; dashed curve, $S_{\text{int}}(t)$ interpolation; points, $1/I_i(T_i)$ values. (c) The values of instantaneous frequency $\bar{\omega}^H(T_i)$ (according to Hilbert) corresponding to the threshold level crossing (black circles connected by dashed line) and the $2\pi/I_i(T_i)$ values (open circles connected by solid line representing the $\omega_{\text{int}}(t)$ function).

Let us first consider the IF model. It was demonstrated [6] that, in the high-frequency approximation, an IF ISI sequence represents a nonlinear transformation of the input signal

$$I_i \approx \theta/S_i, \quad S_i = S(T_i). \tag{2}$$

Since the highest Lyapunov index is invariant with respect to nonlinear transformations, the λ_1 value calculated for the I_i sequence must coincide with the results of calculation using the $S(t)$ signal. Our approach to calculating the Lyapunov index is essentially as follows [10, 11]. Once the IF ISI sequence is known, we may use Eq. (2) to obtain the sequence

$$\frac{1}{I_i} \approx \frac{1}{\theta}S_i = kS(T_i) \tag{3}$$

representing the values of the input signal multiplied by a certain constant k at the fixed time instants T_i. In order to pass to a signal with uniform time scale, the points $1/I_i(T_i)$ are interpolated by a smooth function $S_{\text{int}}(t)$, for example, by a cubic spline. This restores the linear transformation of the input signal to a certain approximation: $S_{\text{int}}(t) \approx kS(t)$ (Fig. 1b). Consequently, the $S_{\text{int}}(t)$ signal would retain both geometric and dynamic characteristics of the attractor corresponding to the input signal (external action). Evidently, this scheme is only valid within a certain approximation. However (see Fig. 2a), the λ_1 value calculated from the $S_{\text{int}}(t)$ signal by a method described in [12] coincides with the results of the λ_1 calculation proceeding directly from $S(t)$. By analogy with [6], the input signal was represented by a linear transformation of the first coordinate of a Rössler system in a chaotic mode: $S(t) = x(t) + C$, $\theta = 35$, $C = 40$.

Raising the threshold level leads to an increase in the average time interval $\bar{I}$ and, hence, a decrease in the accuracy of Eq. (2). Figure 2b shows the results of λ_1 calculation depending on the selection of constant C for the same test system (variation of the C value is equivalent to shifting the threshold level). As seen, the index remains virtually unchanged for $C > 30$. This corresponds to $\bar{I} < T_0/5$–$T_0/4$, where T_0 is the base period of oscillations of the $x(t)$ signal (cf. Fig. 2c). For smaller C values, the size of the time window occupied by the vector of state is greater than the signal correlation time, which hinders reliable determination of the dynamic characteristics of the external action (input signal) [8].

The DS reconstruction from a sequence of recovery times presents a more complicated problem. A possible approach to solving this task, proposed in our previous work [9], is as follows. First, a transition is performed from the time intervals I_i to the points $\omega(T_i) = 2\pi/I_i$ corresponding to the instantaneous frequency values averaged over the I_i intervals. Then, the $\omega(T_i)$ points are interpolated by a smooth function (a cubic spline) $\omega_{\text{int}}(t)$ to pass to a signal with the uniform time scale used in the attractor reconstruction. Using the resulting time dependence, it is possible to describe behavior of the average instantaneous frequency $\bar{\omega}^H(t)$ (Fig. 1c), while the reconstructed attractor retains the dynamic characteristics of chaotic oscillations in $S(t)$ (Fig. 2d). [For the TC model, the $S(t)$ signal was represented by the first coordinate $x(t)$ of the Rössler system.]

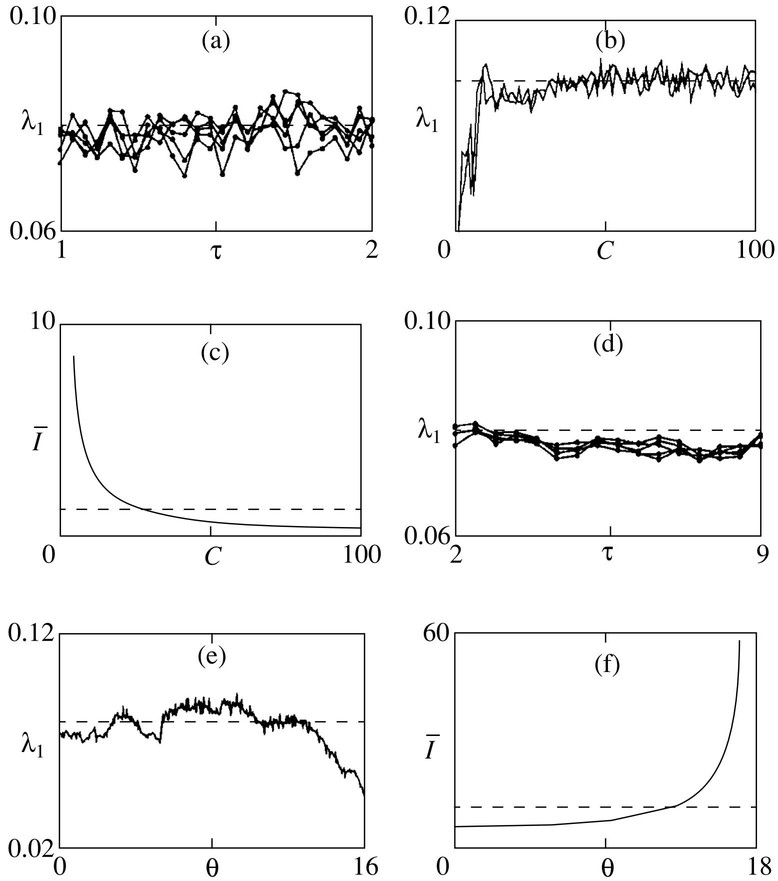

Fig. 2. (a, d) Plots of the highest Lyapunov index versus the delay time τ calculated for various dimensions of the embedding space using the (a) IF ISI and (d) TC ISI sequences for a Rössler system representing a source of chaotic oscillations (dashed lines show the λ_1 values calculated using the given system of equations); (b, e) plots of the highest Lyapunov index versus threshold level Θ calculated for the (b) IF and (e) TC models (in the former case, the threshold variation was modeled by equivalent change in the input signal shift C, see the text); (c, f) plots of the average time interval $\bar{I}$ versus threshold for the (c) IF (Θ modeled by C) and (f) TC models. The dynamic characteristics of the input chaotic oscillations can be determined provided that $\bar{I}$ does not exceed the characteristic time T_c indicated by the dashed line.

By analogy with the IF model, we have studied in detail dependence of the quality of reconstruction on the selection of the threshold level Θ. A shift of this threshold has a clear physical meaning. Indeed, assume that we have changed the input signal amplitude. From the standpoint of the DS theory, this would affect neither the geometry nor dynamics of the chaotic oscillations. At the same time, a change in the amplitude significantly modifies the structure of the TC ISI output sequence (i.e., the ISI distribution function and the recovery time transformation). These changes are so significant that it was previously considered impossible to estimate characteristics of the chaotic signal at large Θ (small-amplitude input signals) [7, 8]. However, as seen from Fig. 2e, the highest Lyapunov index is independent of the threshold, provided that $\bar{I}$ does not exceed a characteristic time scale T_c of the chaotic oscillations (in our case, the time of predictability $T_c \approx 1/\lambda_1$ [13]). Therefore, despite the inability to estimate the input set geometry [8], we may still calculate the dynamic characteristics using the ISI sequence. [Naturally, in speaking of the retained characteristics, we imply the results of approximate numerical experiments rather than of the rigorous mathematical calculation: the characteristics can be evaluated to within $\approx\pm 10\%$.]

The main conclusions from our investigation are as follows. The dynamic characteristics of a signal from small-scale dynamic system entering the neuron input can be determined from the output ISI sequence, provided that the average time interval does not exceed the characteristic time scale T_c. The time scales may be different for various mathematical models of spike generation: for the IF model, the T_c value does not exceed the time required for the correlation function to attain the first zero (for signals with clearly pronounced base frequency in the spectrum, this corresponds to a quarter of the base period); for the TC model, the characteristic time scale is markedly greater and equals approximately to the predictability time. In this work, we presented the results of calculations performed for the Rössler system. The results were confirmed by data of a series of experiments performed with various sources of chaotic oscillations. Thus, the dynamic characteristics of random oscillations determined from the output ISI sequence are retained upon linear transformations of the neuron input signal. For the restrictions formulated above, the accuracy of determining these characteristics is independent of the structure of the output sequence of spikes.

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