Reconstruction of dynamic systems using short signals

A. N. Pavlov, N. B. Yanson, T. Kapitaniak, and V. S. Anishchenko

Saratov State University Technical University, Łódź, Poland

(Submitted January 11, 1999)
Pis'ma Zh. Tekh. Fiz. **25**, 7–13 (June 12, 1999)

It is demonstrated that in principle, a global reconstruction technique can be used to reconstruct a dynamic description from short signals (less than ten base periods of the oscillations), which means that the reconstruction technique can be employed to estimate metric and dynamic characteristics of the operating regimes of dynamic systems using short time series.
 [S1063-7850(99)00206-2]

One method of studying various processes and phenomena in real life involves constructing and investigating mathematical models of them. Such a model is conventionally constructed by allowing for all the most important factors which influence the behavior of the system. The modeling task is complicated considerably if information on the object being studied is limited by a one-dimensional time series of one of the coordinates of state of the system. In 1987, a global reconstruction algorithm was proposed to construct a mathematical model for this case.[1,2] This algorithm can provide a dynamic description in the form of a system of first-order ordinary differential equations or discrete mappings and is implemented in two stages. The first stage involves determining the embedding space dimension and reconstructing the attractor using the scalar time series $a_i = a(i\Delta t),\ i=1,\dots,N$. The second stage involves defining *a priori* the general form of the equations and specifying the evolution operator by the least squares method. This method was later improved[3–5] and new approaches were developed for modeling using a one-dimensional time series.[6,7]

We assume that the system being studied can be described in the form

$$\frac{d\mathbf{x}}{dt}=\mathbf{F}(\mathbf{x}), \quad \mathbf{x}\in R^n, \tag{1}$$

where $\mathbf{F}$ is a nonlinear vector function and $\mathbf{x}$ is the state vector. Several methods are available for reconstructing the vector $\mathbf{x}$ from a time series, of which the following two are the most popular:

$$\mathbf{x}(t)=(a(t),a(t+\tau),\dots,a(t+(n-1)\tau)), \tag{2}$$

$$\mathbf{x}(t)=(a(t),da(t)/dt,\dots,d^{n-1}a(t)/dt^{n-1}). \tag{3}$$

The global reconstruction problem is solved by selecting *a priori* the form of the vector function $\mathbf{F}$ in Eq. (1), calculating the values of $d\mathbf{x}_i/dt$ by numerical differentiation of the scalar time series a_i, and then using the least squares method to specify the evolution operator.

One of the main problems here is selecting the right-hand sides of Eq. (1). Since it is impossible to specify *a priori* even an approximate form of the functions $F_j,\ j=1,\dots n$, these are represented as an expansion in terms of a certain basis, limited to a finite number of terms. In the simplest case, F_j can be defined in terms of ν-degree polynomials:

$$F_j(\mathbf{x}_i)=\sum_{l_1,l_2,\dots,l_n=0}^{\nu} C_{j,l_1,l_2,\dots,l_n}\prod_{k=1}^{n} x_{k,i}^{l_k}, \quad \sum_{k=1}^{n} l_k \leq \nu, \tag{4}$$

where $C_{j,l_1,l_2,\dots,l_n}$ are unknown coefficients which need to be determined. This representation will be used in the present study.

We note that the reconstructed model will be cumbersome and will contain many (usually several tens) of the coefficients $C_{j,l_1,l_2,\dots,l_n}$. The global reconstruction procedure itself, which includes carefully selecting the parameters of the algorithm at all its stages, is very tedious and laborious. When this procedure is implemented, the question arises as to what this model actually gives the researcher in the event of a successful reconstruction. The information of practical interest in the analyses of time series is that on the operating characteristics of the dynamic system generating this time series. Given the implication that in the dynamic system under study there is an attractor, these characteristics are the spectrum of Lyapunov exponents and the dimension. These characteristics are usually calculated using standard algorithms (for instance, Refs. 8 and 9), provided that the time of the series is sufficiently long that the structure of the attractor being studied can be assessed along the phase trajectory. When fundamentally short signals are used (less than ten base periods of the oscillations) it is incorrect to use these methods, because over the observation time the phase trajectory does not have time to visit all the regions of the attractor and/or returns to these regions insufficiently frequently.[10] Here we consider the possibility of using a global reconstruction algorithm to estimate the attractor characteristics in these situations. From this point of view the influence of the signal duration (the number of points N for a fixed discretization step Δt) on the result of the modeling acquires fundamental importance.

In Ref. 11, by applying a reconstruction algorithm to one-dimensional time series obtained by integrating the equations for a Van der Pol oscillator

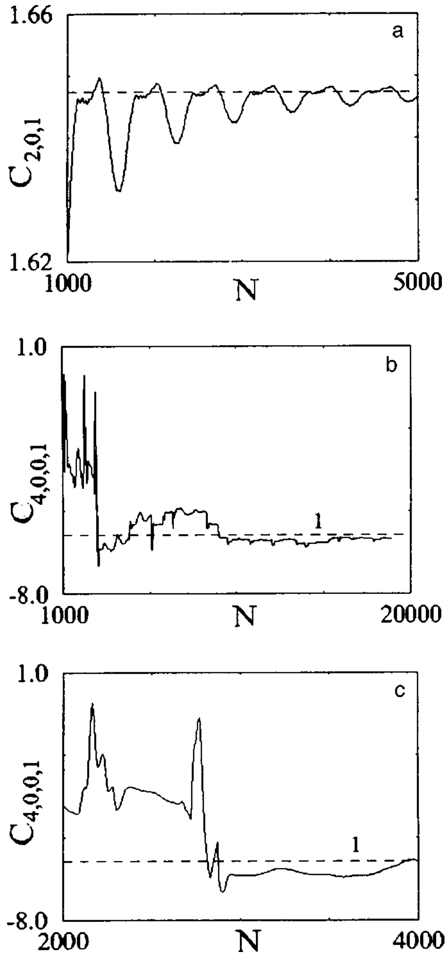

FIG. 1. Arbitrarily selected coefficients in the approximations to the nonlinearities on the right-hand sides of the equations in the reconstructed model as a function of the number of points in the times series: a — Van der Pol oscillator ($n=2$, reconstruction using first coordinate, $\Delta t=0.01$, delay method, $\nu=3$); b, c — for a Rössler system ($n=4$, reconstruction using first coordinate, $\Delta t=0.01$, two coordinates reconstructed by the delay method, one by numerical differentiation, $\nu=3$). Line *1* corresponds to $C^0_{4,0,0,0,1}\approx-5.9$.

$$\frac{dx}{dt}=y, \quad \frac{dy}{dt}=a(1-bx^2)y-x, \quad a=1.0, \quad b=0.3 \quad (5)$$

and a Rössler system

$$\frac{dx}{dt}=-(y+z), \quad \frac{dy}{dt}=x+ay, \quad \frac{dz}{dt}=b+z(x-c),$$

$$a=0.15, \quad b=0.2, \quad c=10.0, \quad (6)$$

and also to various other known model systems, we established that the N dependence of each approximation coefficient $C_{j,l_1,l_2,\ldots,l_n}$ separately exhibits convergence to some limiting value $C^0_{j,l_1,l_2,\ldots,l_n}$ as N increases. Two examples are illustrated in Figs 1a and 1b.

We introduce the relative error in the determination of the coefficient caused by the short length of the time series $\delta_N^{j,l_1,l_2,\ldots,l_n}=|C_{j,l_1,l_2,\ldots,l_n}-C^0_{j,l_1,l_2,\ldots,l_n}|/|C^0_{j,l_1,l_2,\ldots,l_n}|$. Having defined the permissible error $\delta_{\max}^{j,l_1,l_2,\ldots l_n}$, we can determine $N_{\min}$ using the envelope $C_{j,l_1,l_2,\ldots,l_n}$ (N) such that for any $N>N_{\min}$ the value of $\delta_N^{j,l_1,l_2,\ldots,l_n}$ will be less than $\delta_{\max}^{j,l_1,l_2,\ldots,l_n}$. The estimate of $N_{\min}$ indicates the minimum length of the time series required to calculate the approximation coefficients with predefined accuracy.

Note that the behavior of the reconstructed coefficients (convergence) does not depend on how well the *a priori* selected form of the model can describe the operating regime of the initial system. In the following analysis we shall only consider those forms of the right-hand sides for which the solution of the reconstructed system can describe the initial oscillation regime fairly accurately. We also note that solving the problem of reconstructing a system with a periodic regime seems to us fairly trivial. In addition, studying a section of a time series which exhibits only a few oscillations does not allow one to say whether it corresponds to a chaotic or a complex periodic regime. Thus, we shall confine our analysis to the case when the oscillation regime under study is chaotic.

We shall consider the specific example of a Rössler system. For the given values of a, b, and c this system demonstrates chaotic behavior with an attractor characterized by the following spectrum of Lyapunov exponents: $\lambda_1\approx0.09$, $\lambda_2=0.0$, $\lambda_3\approx-10.0$. We shall take the coordinate $x(t)$ discretized with the step $\Delta t=0.01$ as the signal being studied. We shall solve the modeling problem for this signal for various N ($N\in[2000\text{–}4000]$). The other parameters were selected as follows: $n=4$, $\nu=3$. Of the three reconstructed phase coordinates two were obtained by a delay method and one was obtained by numerical differentiation of $x(t)$. Figure 1c shows the dependence of the arbitrarily selected reconstructed coefficient $C_{j,l_1,l_2,\ldots,l_n}$ in this range of N.

In order to obtain a clear representation of the results of the modeling for each reconstructed dynamic system for a selected number of points N, we shall calculate the spectrum of the Lyapunov exponents and the Lyapunov dimension using the Kaplan–Yorke formula.[12] The corresponding dependence of λ_1 and D_L is plotted in Figs. 2a and 2b.

It can be seen from this figure that there is a set of N values for which the attractor of the reconstructed mathematical model (when the other parameters of the numerical system are fixed) has dynamic characteristics similar to those of the attractor of the initial system generating the signal being studied. However, there is also a set of N values for which periodic oscillations ($\lambda_1=0$) are reconstructed instead of dynamic chaos. In Fig. 2a the asterisks also indicate points where the phase trajectory does not belong to the basin of attraction of the attractor of the reconstructed equation (the solutions of the model system do not possess the property of Poisson stability).

In this case, the distribution of the λ_1 and D_L values obtained for various N will have two maxima (Figs. 2c and 2d), one corresponding to the known unsuccessful reconstruction, i.e., the reconstruction of the periodic oscillation regime using a chaotic signal. The second maximum corresponds to $\lambda_1\approx0.08$, $D_L\approx2.016$. In Fig. 2c the dashed lines

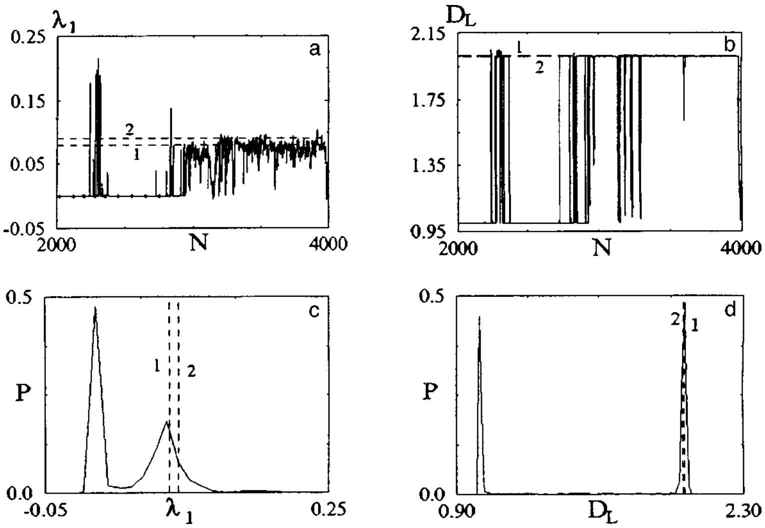

FIG. 2. a, b — Values of the leading Lyapunov exponent calculated by the method described in Ref. 13 and of the Lyapunov dimension of the attractors in the mathematical models reconstructed for each N. The lines *1* give the attractor characteristics of the model system reconstructed using a long signal, i.e., several hundred base periods ($\lambda_1 = 0.08$, $D_L = 2.015$), the lines *2* give the attractor characteristics of the initial system (6) — $\lambda_1 = 0.09$, $D_L = 2.01$; c, d — distribution densities of λ_1 and D_L.

show the values of the leading Lyapunov exponent calculated by a method described in Ref. 13 using the equations of a mathematical model reconstructed by a method of reconstruction using a long signal, i.e., several tens of base periods (line *1*) and using the equations of system (6) (line *2*). The results show that the positive value of the Lyapunov exponent corresponding to the distribution maximum is close to the true value.

To conclude, provided that the *a priori* selected form of the dynamic equations can qualitatively describe the initial chaotic regime, applying the reconstruction algorithm to short time series can give estimates of the characteristics of the initial attractor similar to those which can be calculated by processing long time series using standard methods.

Similar results were obtained for a modified inertial-nonlinearity generator[14] and for various other systems.

This work was partially supported by INTAS Grant No. 96-0305 and by the Royal Society, London.

[1] J. Cremers and A. Hübler, Z. Naturforsch. Teil A **42**, 797 (1987).
[2] J. P. Crutchfield and B. S. McNamara, Complex Syst. **1**, 417 (1987).
[3] G. Gouesbet and J. Maquet, Physica D **58**, 202 (1992).
[4] G. Gouesbet and C. Letellier, Phys. Rev. E **49**, 4955 (1994).
[5] J. L. Breeden and A. Hübler, Phys. Rev. A **42**, 5817 (1990).
[6] H. Voss and J. Kurths, Phys. Lett. A **234**, 336 (1997).
[7] R. Hegger, M. J. Bünner, H. Kantz, and A. Giaquinta, Phys. Rev. Lett. **81**, 558 (1998).
[8] P. Grassberger and I. Procaccia, Phys. Rev. Lett. **50**, 346 (1983).
[9] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, Physica D **16**, 285 (1985).
[10] G. G. Malinetskiĭ, A. B. Potapov, and A. I. Rakhmanov, Preprint No. 10 [in Russian], M. V. Keldysh Institute of Applied Dynamics, Russian Academy of Sciences, Moscow (1993).
[11] A. N. Pavlov, N. B. Yanson, and V. S. Anishchenko, Pis'ma Zh. Tekh. Fiz **23**(8), 7 (1997) [Tech. Phys. Lett. **23**, 297 (1997)].
[12] J. L. Kaplan and J. A. Yorke, Lect. Notes Math. **898**, 204 (1981).
[13] G. Benettin, L. Galgani, and J. M. Strelcun, Phys. Rev. A **41**, 2338 (1976).
[14] V. S. Anishchenko, *Complex Oscillations in Simple Systems* [in Russian], Nauka, Moscow (1990), 311 pp.

Translated by R. M. Durham