Global reconstruction from nonstationary data

N. B. Yanson, A. N. Pavlov, T. Kapitaniak, and V. S. Anishchenko

Saratov State University;
Technical University, Lodz, Poland
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One-dimensional time series of a dynamical system with slowly varying parameters are investigated. For estimation of the characteristics of the attractors of such a system which exist for fixed values of the parameters, it is proposed to ''cut out'' from the time series short segments that belong to the individual attractors and to use them to reconstruct a model dynamical system. © *1999 American Institute of Physics.* [S1063-7850(99)02905-5]

Papers in which dynamical systems are analyzed from one-dimensional time series as a rule have the goal of estimating the characteristics of the operating regimes of the systems, specifically, to calculate the power spectra and moment functions and to determine the geometric and dynamic characteristics of the attractors, etc. One of the most complicated problems is to predict the future behavior of the system and to construct a mathematical model describing the evolution of its state (the problem of global reconstruction).

An algorithm for global reconstruction of a dynamical system from one-dimensional times series was first proposed in 1987.[1,2] In recent years the technique of modeling from experimental data has been discussed in a number of papers: various modifications of the global reconstruction method have been developed,[3–5] and several original approaches to the problem have been implemented.[6,7]

Since most of the algorithms hitherto developed for solving the aforementioned problems are applicable to stationary signals, it is ordinarily assumed that the time series is generated by a finite-dimensional dynamical system with constant parameters,

$$d\mathbf{x}/dt=\mathbf{F}(\mathbf{x},\boldsymbol{\mu}), \quad \mathbf{x}\in R^n, \quad \boldsymbol{\mu}\in R^m, \qquad (1)$$

in which the investigated processes are assumed stationary.

However, if real experimental signals are being analyzed, especially signals of biological origin, such an assumption is not always justified, since the initial objects are open systems, subject to the influence of the surrounding medium. Such systems, on account of the presence of feedback, generally function in a regime of adaptation to changes in the external conditions. The signals generated by them are nonstationary, and the adaptation process can often be interpreted as a variation of the parameters of the system in time.

In this paper we consider the possibility of applying the technique of reconstruction to a one-dimensional time series of a dynamical system with slowly varying parameters for the purpose of determining the dependence of the characteristics of the attractors of the systems on the values of the control parameters.

Suppose that $\boldsymbol{\mu}=\boldsymbol{\mu}(t)$ in the dynamical system (1). Let us make several assumptions under which we will solve the stated problem:

1) the function $\boldsymbol{\mu}(t)$ is oscillatory;

2) for simplicity we restrict consideration to the case of one-parameter modulation, i.e., $\mu_j=\mu_j(t)$, $\mu_k(t)=\text{const}$, $k=1,\ldots,m,k\neq j$;

3) the parameters vary slowly not only in comparison

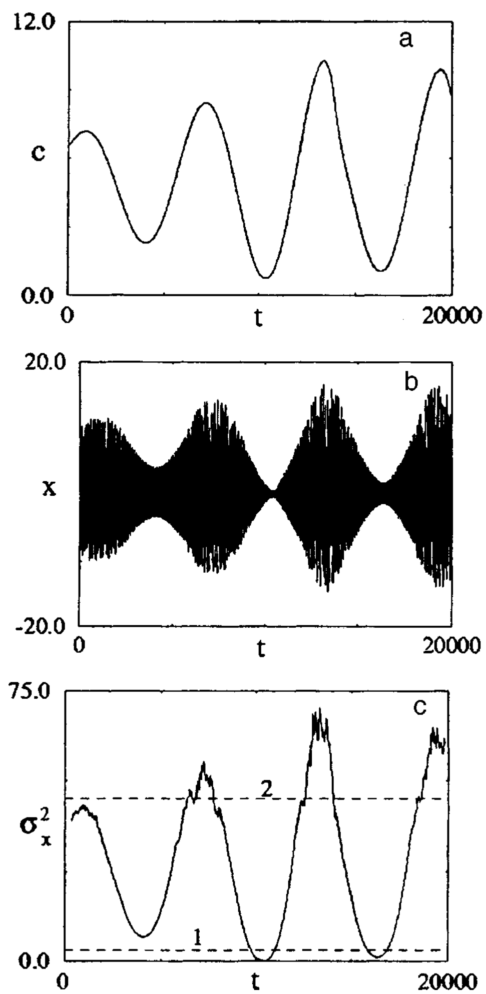

FIG. 1. a: Modulation of the parameter c of the Rössler system; b: corresponding time dependence $x(t)$; c: time dependence of the variance calculated from the signal (b) within a time window that is shifted along the time series.

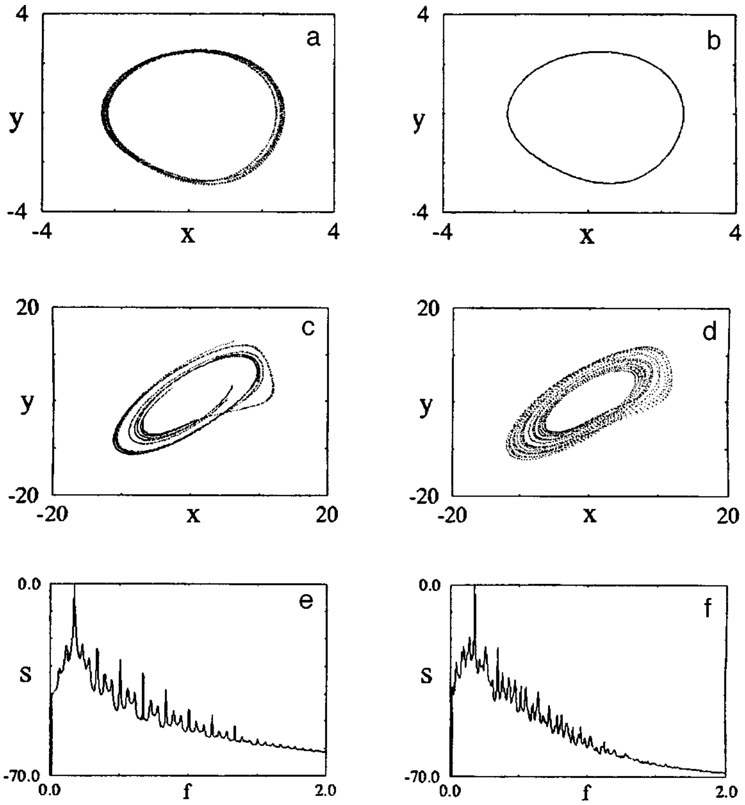

FIG. 2. a,c: Phase trajectories reconstructed from short segments of the time series in Fig. 1b and corresponding to levels *1* and *2* in Fig. 1c; b,d: attractors of the dynamical systems reconstructed from the given phase portrait. The parameters of the reconstruction for Fig. 2 are as follows: a,b — dimension of the embedding space $n=3$, the right-hand sides were approximated by third-degree polynomials, and the coordinates were reconstructed by the method of successive differentiation;[11] c,d — dimension of the embedding space $n=4$, the right-hand sides were approximated by third-degree polynomials here also, but the coordinates were reconstructed by the delay method; e — the power spectrum calculated from the x coordinate of the Rössler system at the value of the parameter corresponding to level *2* in Fig. 1c; f — power spectrum calculated from the solution of the reconstructed model system.

with the period of the base frequency of the oscillations of the system under study but also in comparison with the duration of the transient processes, so that one can neglect the inertial properties of the system;

4) the system does not exhibit multistability phenomena, i.e., for identical values of $\boldsymbol{\mu}$ it functions in the same dynamical regime.

These assumptions allow us to assume that at times when the parameters of the system take on the same values, the phase transitions belong to the same attractor. We will discuss the meaning of the fourth assumption, which may not be necessary, below.

We note that we are analyzing only a time series generated by a system with varying parameters and do not have information about the concrete form of the evolution equations of the system or about the character of the time dependence of $\boldsymbol{\mu}(t)$.

As an example, let us model the given situation with the well-known Rössler model:[8]

$$\frac{dx}{dt}=-(y+z),\quad \frac{dy}{dt}=x+ay,\quad \frac{dz}{dt}=b+z(x-c),$$
$$a=0.15,\quad b=0.2, \tag{2}$$

where the parameter c varies in an irregular way (Fig. 1a) on the interval.[1–12] On this interval a transition occurs from a one-cycle to a regime of chaotic oscillations through a cascade of period-doubling bifurcations. The relation between the mean modulation period of the parameter and the base period of the auto-oscillations $x(t)$ was chosen to be of the order of 1000:1. The observed time series (Fig. 1b) is nonstationary.

We choose the time window for the signal $x(t)$ so as to be long enough to encompass several periods of the observed time series but short enough that the parameters of the system can be considered approximately constant during this time interval. We will determine the character of the parametric modulation. For this we will calculate the moments for segments of the time series inside a time window 50 dimensionless time units long (encompassing approximately 8 periods of the oscillations).[1)] By shifting the window along the signal, we construct the time dependence of these moments. Obviously a changeover of the regime of functioning of the system will not lead to a change of all the moments simultaneously. For example, for the time series in Fig. 1b the mean value does not change as the control parameter is varied. However, one can always find moments which will react to a changeover of regime, and for the time series under study the variance is one of those (Fig. 1c). The behavior of this graph qualitatively reproduces the function $c(t)$ (Fig. 1a).

Let us now consider two levels corresponding to two different values of the variance (Fig. 1c), which, as we assume, correspond to two fixed values of the parameter c. In an actual calculation with real signals we will not know the values of the control parameters of the system and will only assume that each set of them corresponds to some value of the moment. However, for the modeled situation we know that the lower level (line *1*) corresponds to a one-cycle limit cycle, while the upper level (line *2*) corresponds to a chaotic regime.

We choose small neighborhoods of the points (the size of the neighborhood is about one period of the oscillations) in which each chosen level crosses the time dependence of the variance. We ''cut out'' the corresponding segments of the time series for these neighborhoods and apply to them the standard embedding technique, e.g., the delay method or the method of successive differentiation.[9–11] The results of the reconstruction of the phase portrait for such segments for the two different levels are shown in Figs. 2a and 2c, respectively. The segments of the phase trajectories do not ''meet up'' with one another and are rather short, but we assume that they belong to the same attractor (the regular and the chaotic, respectively).

Since the number of state vectors reconstructed by this method may be quite small and, moreover, the reconstructed segments of the phase trajectories do not ''meet up,'' the application of the standard method of signal processing, such as calculation of the autocorrelation function, power spectrum, Lyapunov exponents, etc., to these data is problematical. At the same time, employing the technique of global reconstruction presupposes only knowledge of a set of state vectors at discrete times and their time derivatives (here the length of the signal may be relatively short)[12,13] and does not impose any requirements on the continuity of the phase trajectory. Let us illustrate the application of the given algorithm to the phase portraits shown in Figs. 2a and 2c (the parameters of the algorithm are given in the captions).

Figures 2b and 2d show the attractors corresponding to the reconstructed models. Then, having obtained a dynamical description of the necessary regime, one can by numerical integration generate a phase trajectory of arbitrary duration and calculate from it the characteristics of the attractors by standard algorithms. In particular, the maximum Lyapunov exponent λ_1 calculated for the chaotic attractor of the Rössler system by the method of Ref. 14 for $c \approx 8.0$ and corresponding to level *2* in Fig. 1d, has a value ≈ 0.065. The Lyapunov exponent calculated by the same method from the equations of the model system obtained for level *2* is ≈ 0.052, somewhat less than its ''true'' value. For comparison, in Figs. 2e and 2f we show the power spectra of the initial chaotic attractor of system (2) and of the chaotic attractor of the corresponding reconstructed system.

In summary, with the multiwindow reconstruction procedure discussed in this paper, by shifting the straight lines *1* and *2* in Fig. 1c one can track the evolution of the characteristics of the regimes of functioning of a dynamical system with slowly varying parameters from a one-dimensional time series.

We note in closing that the technique described above is applicable under conditions where the system under study does not exhibit multistability and the related hysteresis. If this is not the case, then the time dependences of the moment functions will not reproduce the law of modulation of the parameters. However, since the moments characterize the regime of oscillations and not the values of the parameters, we assume that this technique will permit reconstruction of the necessary attractors even in that case, but this question will require a separate detailed investigation.

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[1)]To calculate the moment functions of the random process one needs to know its distribution densities. However, by making the assumption that the process under study is stationary over the chosen time segment (we assume that the oscillations occur at an attractor) and ergodic, one can replace the averaging over an ensemble of time series by a time average. For calculations to high accuracy in averaging over time it is necessary that the time series be long. Here, with short time series, we can evaluate the moments only approximately, and that will lead to choppiness of the graphs of their ''time'' dependence (Fig. 1c).

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