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Return times dynamics: role of the Poincaré section in numerical analysis

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Abstract

We study how different measures estimated from return time sequences are sensitive to choice of the Poincaré section in the case of chaotic dynamics. We show that scaling characteristics of point processes are highly dependent on the secant plane. We focus on dynamical properties of a chaotic regime being more stable to displacements of the section than metrical characteristics.

1. Introduction

One of the commonly used methods allowing to study the structure and properties of a chaotic attractor being the solution of some dynamical system consists in the introduction of the Poincaré section. The given approach decreases the dimension of the phase space often making easier further numerical investigations. In general, there exist two possibilities. On the one hand, it is possible to analyze a set of points being the coordinates of successive intersections of the secant plane by a phase space trajectory. On the other hand, a series of time intervals between these intersections can be considered (return times), i.e., all available information about the features of a chaotic regime will be carried by a sequence of event timings only. The last type of processes (so-called point processes [1]) is rather popular in many areas of science (e.g., in neurophysiology) and represents an object of a high interest [2].

Within the frameworks of dynamical systems theory the Poincaré section is introduced in such a way that it will be crossed by all phase space trajectories belonging to a chaotic attractor (the case of *correct* introduction of the section). This situation becomes complicated if an arbitrary choice of the secant plane is out of our abilities: A series of return times may represent, e.g., a process at the output of a threshold device with an external complex driving [3]. For large threshold level or for small amplitude of a forcing signal some loops of the phase space trajectory will be missed (we can speak therefore about the *incorrect* introduction of the Poincaré section). When dealing with return times it may be impossible to determine which of two possibilities takes place and a question arises: What can be said about the features of a chaotic forcing on the basis of point processes analysis in both these cases?

Partly, this question was studied in recent publications [3–6]. Castro and Sauer [3] discussed the dependence of correlation dimension computed from interspike intervals produced by simple neuron models versus the threshold levels. According to their work, dimension calculations from point processes are highly dependent on the details of firing thresholds. Following Hegger and Kantz [4], Sauer's embedding theorem [6] originally proved for integrate-and-fire processes is valid for return times as well. In [7] it was shown that dynamical properties of a chaotic regime (the largest Lyapunov exponent) can be extracted from intersection intervals even if many of the smaller oscillations fail to

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cross the secant plane. At the same time, displacements of this plane could lead to spurious values of the second exponent [8].

In the present work we perform a comparative study of different measures estimated from return times in order to testify their sensitivity to introduction of the section. We show that an increase of the mean intersection interval can lead to appearance of relatively long-range correlations in data being analyzed. We report the results of dynamic entropies estimation and discuss the dependence of the given characteristics on the details of symbolic representation of return time sequences.

2. Correlation dimension and Lyapunov exponents

Let us consider the Rössler system [9]

$$\begin{aligned} \frac{dx}{dt} &= -(y+z), \\ \frac{dy}{dt} &= x + 0.15y, \\ \frac{dz}{dt} &= 0.2 + z(x-10) \end{aligned} \tag{1}$$

as the source of chaotic oscillations and introduce the Poincaré section in the following way: $x = \Theta$, where a constant Θ can take any values in the range $|\Theta| \leqslant 17.2$. The question whether the metrical properties of a chaotic regime can be determined from return times was studied by Castro and Sauer [3]. Using the Takens estimator [10] being a variant of Grassberger and Procaccia algorithm [11] for correlation dimension (D_2) calculation they found that a move of the secant plane creates difficulties: the value of D_2 becomes rather sensitive to numerical parameters such as the embedding dimension, for example. Thus, correlation dimension calculations in practice depend on the choice of Θ although it is supposed [3] that in theory D_2 can be obtained from return times even for *incorrect* introduction of the secant plane.

Additionally to the results of Castro and Sauer [3] note, that displacements of the Poincaré section change significantly the structure of time intervals leading to decrease (or even to disappearance) of the linear segment in double logarithmic plot of the correlation integral versus spatial separation. As a consequence, it may be unclear how to determine the dimension within the frameworks of standard procedure for its calculation [11]. In particular, Fig. 1a shows the values of D_2 estimated from local slopes of the above mentioned ln–ln plot in five-dimensional phase space for three arbitrary chosen sections: $\Theta = 1$, 3 and 9. (We have analyzed sequences consisting of 25,000 return times.) In the case of $\Theta = 9$, even approximate value of correlation dimension is under question in connection with a high sensitivity to choice of the scaling region at the determination of D_2. The results in Fig. 1a are obtained using the delay method for discrete sequences of time intervals. An approach suggested in [12] will increase the value of D_2 by one. However, the problems being discussed remain the same.

Displacements of the secant plane create probably less problems if we need to extract dynamical characteristics [13] though it is necessary to note, that results of numerical calculations strongly depend on the details of used algorithms [7]. A possible way of how to estimate the largest Lyapunov exponent (λ_1) from return times is considered in [12]. It is based on an approximation of the averaged instantaneous frequency. According to the work [7], the value of λ_1 can be calculated in the case of *incorrect* introduction of the Poincaré section if the mean return time does not exceed some temporal scale which corresponds approximately to the prediction time for chaotic oscillations [14] (Fig. 1b). Certainly, when discussing the results of numerical experiments we should mention that the measure being estimated will depend on a variation of algorithmic parameters. In our case, the insensitivity of Lyapunov exponent to displacements of the Poincaré section means that the value of λ_1 computed from time intervals will coincide with the value obtained from original chaotic oscillations $x(t)$ with an accuracy of about 10–15% (dependence 1 in Fig. 1c).

Fig. 1c demonstrates also the results of the second Lyapunov exponent (λ_2) estimation being more sensitive to the value of Θ. True exponent is obtained for secant planes chosen near the equilibrium points of the Rössler attractor (Θ is close to zero). Here, even the case of *correct* introduction of the section ($|\Theta| \leqslant 5.3$) can lead to problems in the determination of λ_2.

3. Scaling features of return times

Let us discuss how a choice of the Poincaré section influences scaling properties of return times. We shall consider for this purpose two approaches, namely, wavelet transform modulus maxima method (WTMM) [15,16] and detrended fluctuation analysis (DFA) [17,18].

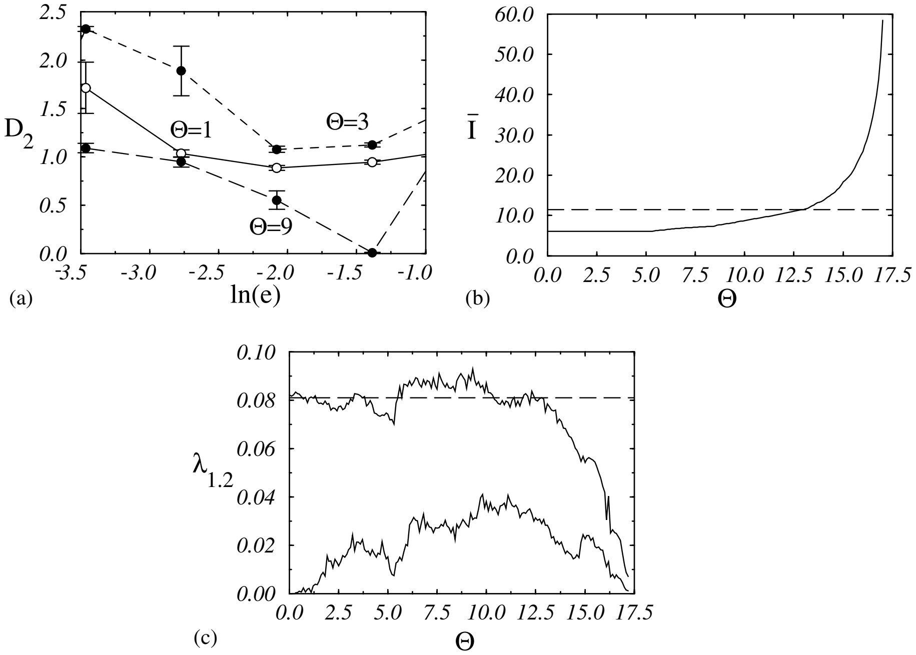

Fig. 1. (a) Correlation dimension estimated using Grassberger and Procaccia algorithm [11] for different scaling regions and for three arbitrary chosen Poincaré sections. The values of D_2 are calculated as local slopes of ln–ln plot of the correlation integral versus spatial separation. (b) Mean return time versus parameter Θ. Dashed line corresponds to the prediction time for chaotic regime. (c) Lyapunov exponents calculated from point processes. Dashed line marks the value λ_1 estimated from original oscillations $x(t)$.

The first approach described in details in [15] supposes the characterization of a time series by means of the function $D(h)$ called *singularity spectrum*, where D is the fractal dimension of the subset of data characterized by the Hölder exponent h. In this method, a study of the local singular behaviour of time series is based on calculations of the wavelet transform coefficients. The WTMM-algorithm assumes the building of a partition function $Z(q,a)$ being the sum of the qth powers of the local maxima of modulus of the given coefficients at some scale a and an analysis of the power-law behaviour: $Z(q,a) \sim a^{\tau(q)}$. Variations of powers q allow to obtain a number of Hölder exponents $h = \mathrm{d}\tau(q)/\mathrm{d}q$ and estimate the Hausdorff dimensions D of the corresponding subsets of data using the Legendre transform [15]. The values of h characterize a power-law scaling behaviour of the wavelet transform coefficients along maxima lines [19].

As applied the WTMM-method to return times we can see (Fig. 2a) that a move of the secant plane results in changes of Hölder exponents. (The shape of singularity spectrum and the values of D are less sensitive to the choice of the section.) An increase of Θ bring Hölder exponents closer to the level $h = 0.5$ which corresponds to uncorrelated behaviour of time intervals. Strong anti-correlations taking place for $\Theta = 1$ (the mean value of h is approximately equal to 0.05) become weaker for larger Θ. Fig. 2b shows that even a small enough displacement of the section leads to significant changes of the scaling features of point processes. For comparison, the results obtained for two values of Θ are given in this figure ($\Theta = 1$ and $\Theta = 3$, both corresponding to the case of *correct* introduction of the secant plane).

The more is Θ the more oscillations are missed, therefore, the mean intersection interval increases (Fig. 1b). However, the dependence of Hölder exponents on Θ is not a simple growth. In some ranges of the given parameter an increase of the mean return time destroys long-range correlations in time series. Within the limits of other regions of Θ the opposite effect can be obtained (Fig. 2c). For example, numerical values of h estimated for $\Theta = 11$ indicate that the fluctuations in return times exhibit correlated behaviour. Other two dependencies in Fig. 2c ($\Theta = 9$ and 15) correspond to uncorrelated behaviour. These results show that the Hölder exponents (and therefore the effect of long-range correlations in point processes) strongly depend on the choice of the section.

Similar conclusions follow from detrended fluctuation analysis [17,18]. In DFA technique, a single quantity α is usually estimated being a scaling exponent of the power-law dependence: $F(n) \sim n^{\alpha}$, where F is the root-mean-square fluctuation of an integrated and detrended time series and n is the window size used for linear fit of the local trend. In practice, α can be found as the slope of the line relating $\ln F$ to $\ln n$ [17]. Slopes may not coincide for different scaling regions. That is why an analysis of local exponents may be performed to characterize the details of the complex

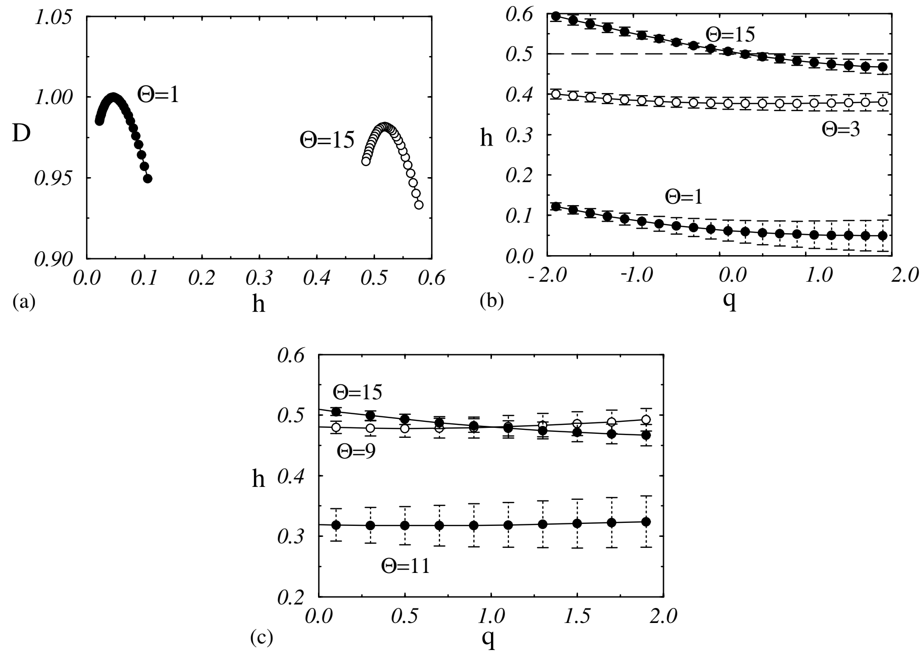

Fig. 2. (a) Singularity spectra of WTMM-method for $\Theta = 1$ and 15. (b) and (c) Hölder exponents versus parameter q corresponding to different secant planes.

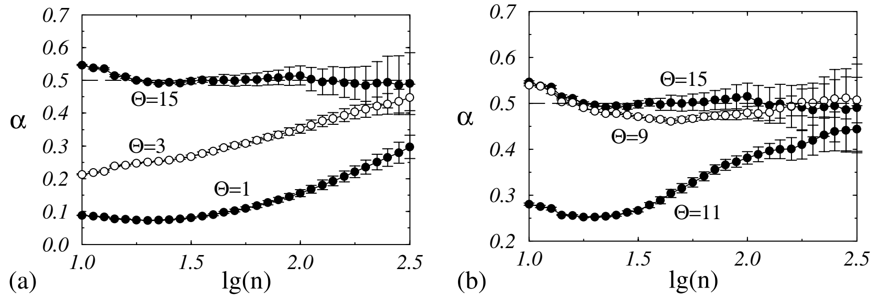

Fig. 3. Local scaling exponents of DFA-algorithm.

structure of time series. Like Hölder exponents of WTMM-method, the numerical values of α indicate the presence of correlations of different types if $\alpha \neq 0.5$ and uncorrelated behaviour for $\alpha = 0.5$.

Again, we can see that displacements of the Poincaré section change significantly scaling properties of return times (Fig. 3a). Here, local scaling exponents are highly sensitive to the window size. In the case of $\Theta = 3$, strong enough correlations existing in wide range of scales become weaker for large n (α approaches 0.5). Fig. 3b testify that within the limits of some regions of Θ an increase of the mean intersection interval can lead to the appearance of correlations in point processes (the case of $\Theta = 11$ in comparison with $\Theta = 9$ and 15). Thus, using two different approaches we get sure that scaling features of return times are highly dependent on the choice of the Poincaré section as in the case of its *correct* introduction ($|\ \Theta\ | \leqslant 5.3$) as when some loops of the phase space trajectory fail to cross the secant plane.

4. Entropy analysis

In this Section we focus on the entropy analysis of point processes. Unlike other approaches (e.g., correlation dimension or Lyapunov exponents calculation) the given analysis requires an initial transformation of a time series into a symbolic sequence [20]. If such a transformation is realized we may introduce the entropy per block of length n (the n-gram entropy):

$$H_n = -\sum p^{(n)}(A_1, \dots, A_n) \log p^{(n)}(A_1, \dots, A_n) \tag{2}$$

and conditional entropies (or n-gram dynamic entropies): $h_n = H_{n+1} - H_n$, where $A_1, \dots, A_n$ are the letters of some block and $p^{(n)}(A_1, \dots, A_n)$ is the probability to find this block at an arbitrary but fixed position in symbolic string [20]. The limit $h = \lim_{n\to\infty} h_n$ is called *entropy of the source*. In this paper, logarithms are taken in k-units, where k is the length of an alphabet, i.e., the number of different possible letters. Numerical values of h_n depend on the details of the above mentioned transformation.

In our study return times are considered as a discrete sequence of real numbers. The partitioning of a time series is provided in such a way that the probabilities of all k symbols approximately coincide. Fig. 4a shows the results of h_5 estimation in dependence on the parameter Θ for $k = 3$, 4 and 5. (Sequences being analyzed consist now of 100,000 symbols.) The conditional entropy normalized by the mean intersection interval decreases however this decrease is slow enough for wide range of Θ including partly a region of *incorrect* introduction of the Poincaré section. Similar results can be obtained also for other block lengths. In analogy with the largest Lyapunov exponent dynamic entropies are less sensitive to displacements of the secant plane than fractal dimensions or scaling characteristics of point processes. This property of dynamic entropies may be important in practice if we aim to provide some classification of a chaotic forcing.

A binary representation of real value data (the case of $k = 2$) leads to more complex dependence of conditional entropies h_5 versus Θ (Fig. 4b). The same effect can be obtained when using other techniques. In particular, we can consider an approach of Lempel–Ziv complexity [21] that also allows to estimate the entropy of the source h (Fig. 4c). Characteristics calculated by means of these two approaches (Fig. 4b and c) are rather sensitive to displacements of the secant plane.

Taking other lengths of an alphabet for the estimation of conditional entropies ($k > 2$) we have obtained the values of h_5 being relatively less than the corresponding results for $k = 2$ (Fig. 4a). The more is k the less are measures under study, i.e., the details of symbolic representation of time series influence our calculations (probably, in connection with finite sample effects). We get sure that the dependence $h_5(\Theta)$ becomes less sensitive on Θ as k increases. The values h_5 are rather close to the largest Lyapunov exponent (λ_1).

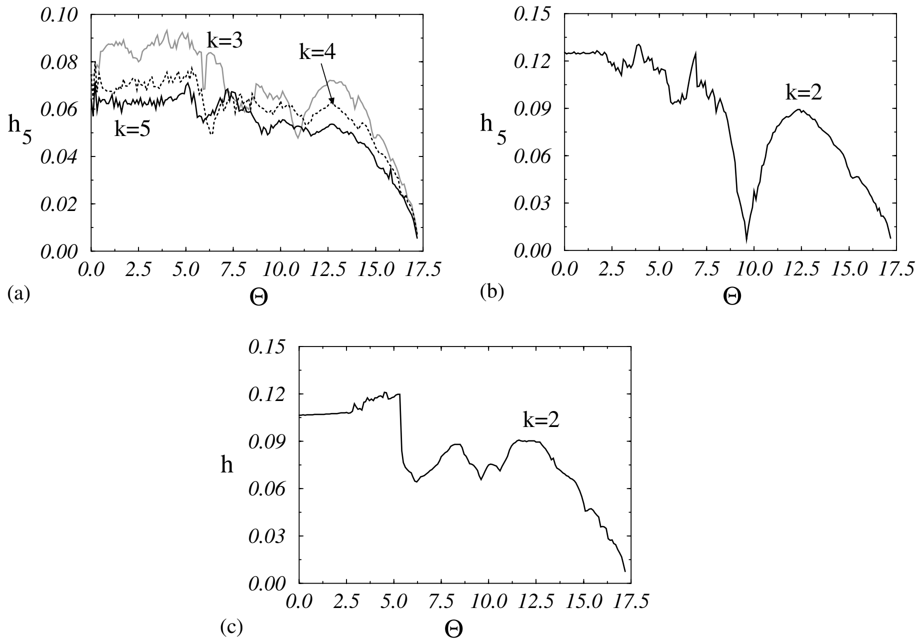

Fig. 4. (a) and (b) The dependencies of conditional entropy h_5 on Θ for different lengths of an alphabet k. Here, the values h_5 were normalized by the mean intersection interval $\bar{I}$ (i.e., we consider $h_5/\bar{I}$). (c) Entropy of the source estimated by means of Lempel–Ziv approach for binary representation of return time sequences ($k = 2$).

For small Θ, conditional entropies fluctuate concerning some mean values. These fluctuations may be reduced when considering longer sequences of return times. When processing finite amount of data an accuracy of numerical calculations becomes of a high importance. In the case of $k = 5$ fluctuations of h_5 do not exceed 15% if in analogy with Lyapunov exponents we shall limit ourself by some region of Θ, e.g. $|\Theta| < 10$. Therefore, we can speak about the insensitivity of measures being estimated to displacements of the Poincaré section within the range of the given accuracy (if missings of the small oscillations occur not too often).

5. Conclusions

In our work we discussed the sensitivity of different measures computed from return times to choice of the Poincaré section in the case of chaotic dynamics. The main results of this study consist in the following.

Displacements of the section change significantly scaling properties of return times and relatively long-range correlations in point processes. An increase of Θ means that more and more oscillations will be missed. However, in the case of *incorrect* introduction of the section correlations exist within some regions of Θ and disappear within other regions.

We can conclude that dynamical properties of a chaotic regime (and, therefore, measures characterizing these properties) are more stable to displacements of the secant plane than metrical characteristics. That is why when speaking about a threshold device with an external chaotic driving the estimation of Lyapunov exponents or entropy analysis give us the possibility to characterize the features of a forcing regime from point processes independently (under quite general conditions) on the amplitude of chaotic oscillations or details of threshold levels.

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