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Reconstruction of dynamical and geometrical properties of chaotic attractors from threshold-crossing interspike intervals

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We reconstruct the largest Lyapunov exponent and fractal dimension of a chaotic attractor using threshold-crossing interspike intervals alone. We show that in certain cases one may reconstruct from this data a set looking very similar to the initial attractor. We also give an explanation of this possibility based on the concept of instantaneous frequency. [S1063-651X(98)51207-X]

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The problem of extracting information about a dynamical system from experimental data is of great importance for all those who study the systems of natural origin that may be thought of as black boxes. The traditional way of obtaining experimental data is to fix the sampling step Δt and store the values of some state variable $s(t)$ corresponding to the time moments $i\Delta t$. The fundamental works [1] state that the delay method applied to such time series often allows one to reconstruct a set being topologically equivalent to the initial system attractor. On the base of these works a series of methods have been developed allowing one to evaluate attractor dimension [2] and the largest Lyapunov characteristic exponent (LCE) [3], to detect unstable equilibrium points [4] and unstable periodic orbits [5], and, finally, to create in some cases global models [6] or different predictors [7].

Another way of getting experimental data which is often preferred by biologists is to generate spike-train data [8]. One imposes a certain condition on the value of state variable $s(t)$ and records the interspike intervals (ISI) T_i between the time moments for which the condition is fulfilled. There are two general methods for recording spike-train data. The first one is to generate a spike when $s(t)$ crosses some threshold level θ in one direction, and measure ISI [as an alternative, one may measure the time intervals between the local maxima of $s(t)$] [Fig. 1(a)]. The values of T_i can be treated as the times of trajectory return to the surface defined by the condition imposed to the initial $s(t)$. [The condition $s(t)=\theta$ means the definition of a plane in the phase space, while the condition $\dot{s}(t)=0$, $s(t)$, being a realization $x_j(t)$ of a dynamical system $d\vec{x}/dt=\vec{f}(\vec{x})$, means the definition of a surface $f_j(\vec{x})=0$.]Using this data one is able to compute probability density of return times [or the interspike interval histogram (ISIH)] and its different functionals, e.g., mean value, variance, entropy, etc. Until recently the meaning of the return times was not quite clear. In the works [9,10] it was stated that in certain cases the time delay plot of return times can be a rough analogy of Poincarè map, the latter conclusion being proved by computation of the corresponding correlation dimensions.

The second method to generate spike train data is the use of integrate-and-fire model or other neuron models [11] when the ISI are generated recursively by $\int_{t_i}^{t_{i+1}} s(t)dt=\theta$, where θ is a fixed threshold, t_i are the time moments when spikes occur. The works [9] establish the possibility to reconstruct the geometrical properties of original attractor using the latter method for obtaining spike train data. However, up to now it seemed to be impossible to reconstruct the initial attractor using threshold-crossing interspike intervals alone [9].

The goal of the present Rapid Communication is to demonstrate the possibility to reconstruct a set being very similar to initial chaotic attractor as well as its dimension and largest Lyapunov exponent using only the sequences of threshold-crossing time intervals T_i or the time intervals between the successive maxima (or minima) of initial time series. We show that the results are qualitatively the same for both cases.

To reveal the meaning of threshold-crossing ISI consider the concept of instantaneous frequency. In Ref. [12] three ways of how to introduce the phase of chaotic oscillations are described. The first two ways are connected with the existence of a projection of system's attractor on the plane (x,y) reminding a smeared limit cycle. If this projection exists then one is able to introduce the Poincarè secant so that it would pass through an equilibrium point of the system around which the motion occurs. According to the *first definition*, the phase is defined via the time moments of trajectory's crossings of the secant surface t_i as follows:

$$\varphi^i(t)=2\pi\frac{t-t_i}{t_{i+1}-t_i}+2\pi i, \quad t_i\leq t<t_{i+1}. \tag{1}$$

The *second definition* operates with the above mentioned projection, and the phase is introduced as

$$\varphi^p=\arctan(y/x). \tag{2}$$

The two phases φ^i and φ^p do not coincide, and only the mean frequency defined as the average of $d\varphi^p/dt$ over large time coincides with $2\pi/T$, where T is the average return time. The *third definition* is given by the general approach which is based on the analytic signal concept introduced by Gabor [13]. The analytic signal $z(t)$ is a complex function of time constructed as

$$z(t)=s(t)+is^H(t)=A(t)e^{i\varphi^H(t)}, \tag{3}$$

where $s^H(t)$ is Hilbert transform of initial signal $s(t)$,

$$s^H(t)=\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{s(\tau)}{t-\tau}d\tau. \quad (4)$$

P means that the integral is taken in the sense of Cauchy principal value. Instantaneous amplitude $A(t)$ and phase $\varphi^H(t)$ of $s(t)$ are uniquely defined from Eq. (4), as well as instantaneous frequency being the derivative of $\varphi^H(t)$.

Note that consideration of instantaneous amplitude and phase (or frequency) instead of s and s^H means substitution of variables which is smooth everywhere except the origin. The latter is direct definition of topological equivalence which is valid if the trajectory does not cross the origin. Therefore, one could use either instantaneous amplitude or frequency to reconstruct the original attractor. Thus, we can unambiguously pass to the phase plane $(A,\omega^H),\omega^H(t)=\dot{\varphi}^H(t)$, treat $\omega^H(t)$ as an independent phase variable and use it to reconstruct the original attractor.

Now we turn to experimental data measured as time intervals between intersections by a realization $s(t)$ of some threshold level θ. Suppose that the plane defined as $s(t)=\theta$ can be treated as Poincarè secant. Then we can use the first method to introduce the instantaneous phase φ^i and frequency ω^i, i.e., we can attribute to each time moment t_i the value of $\omega^i(t_i)=2\pi/T_i$. The values $\omega^i(t_i)$ can be qualitatively treated as the points of coordinate reconstructed from $\omega^H(t)$ by means of an averaging method with varying window and known only at discrete time moments t_i. We do not know for sure how the averaged instantaneous frequency $\omega^H(t)$ behaves itself between the time moments t_i, but we can suppose that it is not constant in real chaotic systems and varies smoothly. That is why the natural step is to connect the existing points with some smooth curve to get an idea of how the real frequency averaged over a moving window with varying length would behave, at least qualitatively. By all means, we would never obtain exactly the true dependence, but we hope that the time series obtained in such a way could in certain cases reproduce qualitatively the behavior of one of system's coordinates and thus allow us to reconstruct approximately the view of original attractor and its dynamical and geometrical properties.

As an example, consider the famous Rössler system [14]

$$\dot{x}=-y-z, \quad \dot{y}=x+ay, \quad \dot{z}=b+(x-c)z, \quad (5)$$

in the regime of weak chaos $a=0.15$, $b=0.2$, $c=6.5$. For this regime phase may be introduced by all the three methods [12]. Instantaneous frequency ω^H versus t is shown in Fig. 1(b). Fast variations of $\omega^H(t)$ can be smoothed by means of averaging over the quasiperiod [see bold line in Fig. 1(b)]. In Fig. 1(c) stars mark the points $\omega^i(t_i)$ and circles mark the values $\langle\omega^H\rangle(t_i)=1/T_i\int_{t_i}^{t_{i+1}}\omega^H(t)dt$. The stars are connected by a smooth curve, and the circles by the pieces of straight line. Note that the behavior of these two dependences is very similar. The phase portrait reconstructed by means of Hilbert transform is shown in Fig. 2(a). The same phase portrait in coordinates (A,ω^H) is given in Fig. 2(b). Next, we build the dependence $\omega^i(t_i)=2\pi/T_i$ for the values of T_i measured as time intervals between the coordinate $x(t)$ crossings of zero level and lead the smooth curve through all its points by using interpolation technique [Fig. 1(c)]. Thus, a system's coordinate is reconstructed from threshold-crossing interspike intervals. The phase portrait obtained by delay method from the reconstructed coordinate is shown in Fig. 2(c).

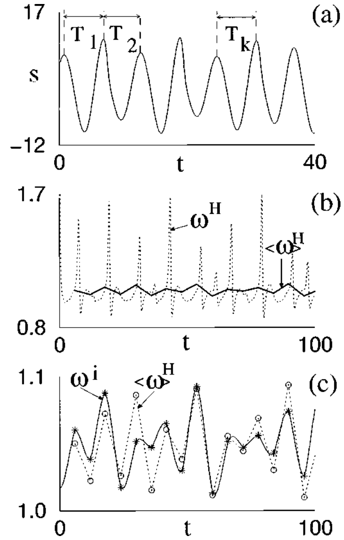

FIG. 1. (a) First coordinate of Rössler system in the regime of weak chaos; (b) time dependence of instantaneous frequency ω^H obtained from the Hilbert transform and the result of averaging $\langle\omega^H\rangle$; (c) the points $\omega^i(t_i)$ connected by the solid curve and the values $\langle\omega^H\rangle(t_i)$ connected by the dashed line. The comments are given in the text.

We only see that it is obviously chaotic. Another way of storing the values of T_i as the time intervals between the local maxima lead to a very similar phase portrait. The box-counting algorithm [2] was used to compute fractal dimensions of the initial Rössler attractor and the sets reconstructed

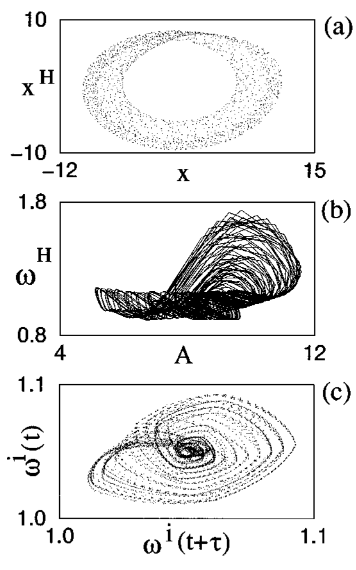

FIG. 2. Phase portraits on the planes (a) (x,x^H), where x is the first coordinate of the system (5), x^H is Hilbert transform of x; (b) (A,ω^H), where A and ω^H are the instantaneous amplitude and frequency, respectively; (c) $[\omega^i(t),\omega^i(t+\tau)]$, where $\omega^i(t)$ is a time dependence obtained by interpolation (the comments are given in the text).

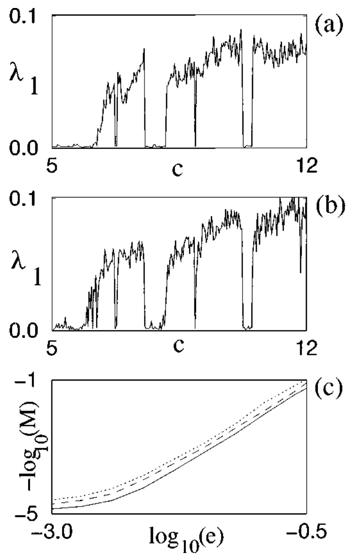

FIG. 3. (a) Largest Lyapunov exponent calculated from x coordinate of the system (5) vs c; (b) largest Lyapunov exponent calculated from $\omega^i(t)$ of the system (5) vs c; (c) plot for estimation of Hausdorff dimension: M is the number of nonempty boxes, ϵ is the size of box. The solid line is obtained from x coordinate of the system (5), the dashed line is obtained from zero-crossing ISI, and the dotted line is obtained from the time intervals between successive maxima of x.

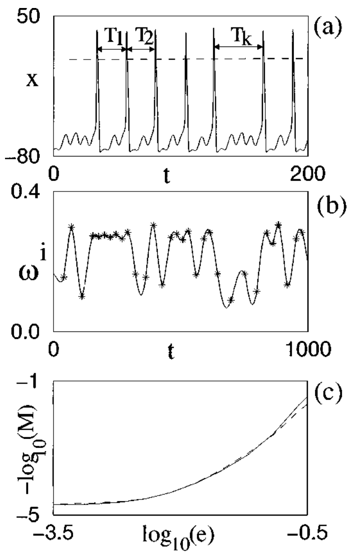

FIG. 4. (a) Action potential coordinate of Hodgkin-Huxley system (the dashed line indicates the threshold level); (b) interpolated time dependence of ω^i; (c) plot for estimation of the Hausdorff dimension. The solid line is obtained from the action potential coordinate of Hodgkin-Huxley system, and the dashed line is obtained from the realization in (b).

from the dependences obtained by means of interpolation from the ISI measured by two ways. The plots of $\log_{10} M$ versus $\log_{10} \epsilon$, where M is the number of nonempty boxes and ϵ is the size of a box, are given in Fig. 3(c). We see that the linear segments of all three graphs are parallel to each other. Now vary the parameter c of Rössler system (7) in the range [5;12] and compute the largest LCE from the coordinate $x(t)$ of this system and from the dependence obtained by means of interpolation technique from threshold-crossing interspike intervals. The corresponding plots are given in Figs. 3(a) and 3(b) and show the remarkable coincidence of quantitative values of Lyapunov exponents in the whole range of parameter values.

The results are qualitatively the same if we assume that the values of ω^i are known not at the time moments t_i but at the time moments iT where $T = 1/n\Sigma_{i=1}^{n} T_i$ being the average return time. Namely, the computed values of dimension and largest Lyapunov exponent preserve under this operation.

Usually the biologists introduce threshold level so that it would allow one to capture only global temporal scales of the system, as in Fig. 4(a). The secant plane defined by such a way is not a Poincarè secant and the definition of phase via the times between crossings of this level appears to be not strict comparing to the first definition of phase. It is obvious that the times T_i introduced for such level are unable to contain detailed information about the system's dynamics. We shall further show that in spite of this fact one is able to extract qualitative information about the dynamical and geometrical properties of underlying attractor.

Consider the Hodgkin-Huxley system describing the neuron activity [15]. To obtain chaotic regime we add periodic current with the amplitude 4 μA/cm^2 and the frequency 0.13 kHz [16]. The equations were numerically integrated to obtain a realization shown in Fig. 4(a). A threshold level was introduced as shown in the figure, and time intervals between crossings of this level were stored. The same procedure of interpolation was performed [Fig. 4(b)] and the value of largest LCE was evaluated from the initial realization ($\lambda_1 \approx 0.025$) and the reconstructed one ($\lambda_1 \approx 0.03$). The plots for computation of fractal dimension were compared for these two realizations [Fig. 4(c)], and although the existence of linear segments is doubtful they go in parallel. This result testifies that in spite of the fact that the condition under which the phase can be defined in the first way is violated, there is an opportunity to obtain a qualitative evaluation of the value of fractal dimension and largest Lyapunov exponent. With this, the phase portrait obtained from the reconstructed realization does not remind the initial one at all.

We state that threshold-crossing interspike intervals are enough to reconstruct fractal dimension and largest Lyapunov exponent of a chaotic attractor of at least saddle-focus type, and also to obtain a set looking similar to the original attractor. This result allows one to apply the algorithms for global reconstruction and prediction to the time series obtained in such a way from spike train data measured from different systems. The suggested technique of interpolation may be disputable, but it demonstrated its workability and allowed us to use the standard algorithms for time series analysis without any modifications. The presented method widens greatly the abilities of those who use only threshold-crossing spike train data to study real systems of different origin, including biological ones.

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