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ONE METHOD FOR RESTORING INHOMOGENEOUS ATTRACTORS

NATALIA B. JANSON,* ALEXEY N. PAVLOV and VADIM S. ANISHCHENKO
Nonlinear Dynamics Laboratory, Department of Physics,
Saratov State University, 83, Astrakhanskaya Street,
410026, Saratov, Russia

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Several methods of restoration of phase portraits were applied to real experimental realizations $a(t)$ of biological origin. The algorithms for global reconstruction were used to create qualitative models of the regimes under study. The results of global modeling were satisfactory for the time series of simple shape, but in case of complicated inhomogeneous realizations the traditional algorithms did not give reasonable models. We suggest a method for restoration of inhomogeneous attractors on $a(t)$ as follows:

$$x_1(t) = \int_0^t a(t)dt, \quad x_2(t) = a(t)$$

while the other coordinates could be restored by any known methods (delay, differentiation, etc.). Such a representation of the attractor's coordinates preserves a simple form of the first equation of the system of differential equations sought

$$\frac{dx_1}{dt} = x_2 \, .$$

This method was tested first on an artificially produced inhomogeneous realization containing pieces with very slow and very quick motion. After that it was successfully applied to real biological inhomogenous realizations.

1. Introduction

At present a large number of methods for restoration of phase portraits on one-dimensional experimental realizations of dynamical systems (DS) have been developed (a good review of them is given in [Breeden & Packard, 1994]). Very often the problem of attractor restoration is stated in connection with the problem of global modeling (reconstruction) of DS on experimental data, i.e. obtaining an explicit form of ordinary differential equations (ODE) or discrete equations qualitatively describing the behavior of systems under study [Cremers & Hübler, 1987; Gouesbet & Letellier, 1994]. Moreover, in recent years the interest to the modeling of systems of biological origin has been growing (e.g. [Rosenblum & Kurths, 1995; Kremliovsky *et al.*, 1996; Baier *et al.*, 1993]).

Among the systems of biological origin one may often encounter the ones whose realizations are highly inhomogeneous, i.e. containing the segments with quick motion followed by the segments with very slow motion. A typical example of such time series is an electrocardiogramme of a human heart to whose thorough exploration much attention is being paid during recent years, e.g. [Babloyantz & Destexhe, 1988; Saparin *et al.*, 1996].

*E-mail: nata@chaos.ssu.runnet.ru

As it is preferable to obtain models having as simple form as possible, the method of successive differentiation of initial time series is rather popular as it leads to the ODE's of the following form:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \ldots, \qquad \dot{x}_N = f(x_1, x_2, \ldots, x_N). \tag{1}$$

But besides the problem of noise greatly affecting the process of derivative computation the problem of inhomogeneity of the obtained phase portrait arises. Really, the phase coordinates restored by successive differentiation of initial inhomogeneous time series are becoming more and more inhomogeneous, and the resulting attractor is a highly inhomogeneous set in the reconstructed phase space, the procedure of global reconstruction being complicated due to this. The other commonly used methods do not significantly improve the situation. For example, when using the famous delay method one may vary the value of delay so that the restored attractor becomes homogeneous enough but the disadvantage of this is the loss of smoothness of the obtained attractor (this will be illustrated in Sec. 4.2).

In the present paper we discuss a very simple method allowing one to solve the problem of sharp inhomogeneity of experimental data when attempting to make a global model of the system under study.

2. Description of Method

Consider a typical experimental realization $a(t)$ which can often be qualitatively presented as a sum of four items:

$$a(t) = O(t) + S(t) + N(t) + C, \tag{2}$$

where $O(t)$ is an oscillatory term which can be expanded into Fourier series and written as $O(t) = \sum_{k=1}^{M}(a_k \sin(kwt) + b_k \cos(kwt))$ (without the constant term corresponding to k=0), $S(t)$ is the "floating" of the average level which may be caused by nonstationarity as well; $N(t)$ is the additive noise whose variance D_N is much less than the variance D_O of a dynamical component $O(t)$ of the process; C is a constant shift of the whole realization.

Consider the integral of the whole realization $a_1(t) = \int_0^t a(\tau)d\tau$ as one of the reconstructed coordinates. For *stationary noiseless* realization with zero average the new variable $a_1(t)$

$$a_1(t) = \int_0^t a(\tau)d\tau = \int_0^t O(\tau)d\tau = O_1(t) \tag{3}$$

consists of only an oscillatory component. Thus, since we are mostly interested in the stationary part $O(t)$ of an oscillatory process, the coordinate $a_1(t)$ will preserve the full information about it. With this, as one operates with an inhomogeneous realization, integration of a "slow" segment will give a quicker time dependence while integration of a "quick" segment will give a slower function of time.

The latter means that $a_1(t)$ will be more homogeneous than $a(t)$, and the attractor restored as follows:

$$\mathbf{x} = \left\{\int_0^t a(\tau)d\tau;\ a(t);\ \frac{da(t)}{dt};\ \ldots;\ \frac{d^{N-2}a(t)}{dt^{N-2}}\right\} \tag{4}$$

or

$$\mathbf{x} = \left\{\int_0^t a(\tau)d\tau;\ a(t);\ a(t+\tau);\ a(t+(N-2)\tau)\right\} \tag{5}$$

will be more homogeneous than when using only the method of delay or differentiation. Moreover, as one uses the embedding (4) the reconstructed ODEs have a simple form (1).

3. Testing on a Simple Model

Let us illustrate the work of the described method using as an example a famous Van der Pol system

$$\dot{x} = y, \quad \dot{y} = a(1 - bx^2)y - x. \tag{6}$$

As $a > 0$, $b > 0$ the limit cycle is the only attractor of the system [Neymark & Landa, 1987]. To obtain an inhomogeneous realization (alternation of "pauses" and quick segments) we perform a nonlinear transformation of coordinate $x(t)$ obtained by numerical integration of Eqs. (6) (as $a = 1$, $b = 1$). Conventionally speaking, signal $x(t)$ passes a rectifier with a given nonlinear characterictic. As a result, the obtained realization $a(t)$ has the required shape as shown in Fig. 1(a), and after subtracting the average value from it (so that the average value of the resulting realization becomes zero), it was used for the attractor restoration by method (4) [Fig. 1(b)]. Further, by the method for global fitting of a model described in [Cremers & Hübler, 1987; Anosov *et al.*, 1995; Janson & Anishchenko, 1995], a four-dimensional DS was reconstructed

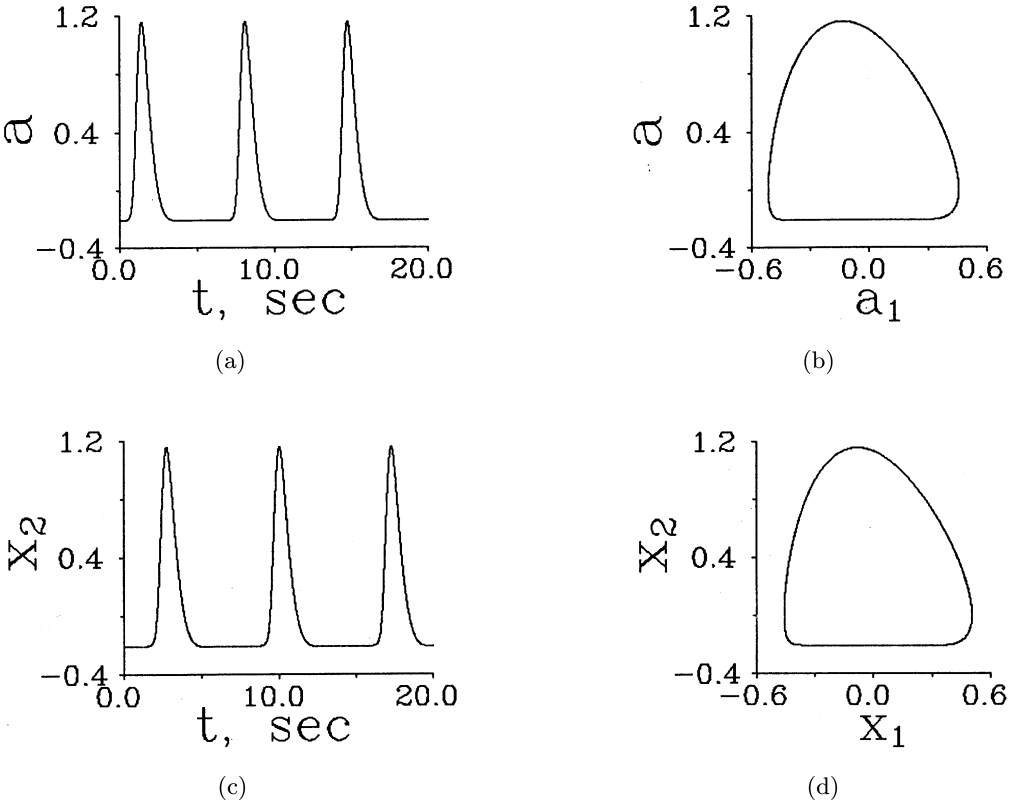


Fig. 1. (a) Initial realization of system (6) after nonlinear transformation by the method mentioned in the text; (b) projection of phase portrait restored on this realization by the method (4); (c), (d) the realization and the projection of attractor of the corresponding reconstructed DS.

possessing the realization $a(t)$ and the attractor shown in Figs. 1(c) and 1(d), respectively. Note, that our attempts to use the methods of delays or derivatives for the creation of global model of such a regime failed.

The artificial technique used for obtaining the realization with "pauses" was neccessary only to test the method of integration embedding and further global modeling.

But in real life such dependences may be encountered rather often. In fact, all the realizations of activities of different hearts are highly inhomogeneous.

4. Global Reconstruction on Real Systems' Realizations

4.1. *Real system: Isolated frog's heart*

First, we apply the integration embedding method to the time series measured from a real system of biological origin, namely the isolated frog's heart (IFH). The experimental realization is the time dependence of a coordinate of point on the surface of the isolated frog's heart contracting in a special solution. This realization has a rather simple shape [Fig. 2(a)] but is inhomogeneous at the same time because of the existence of "pauses" in it. The additional exploration of the measured realization (computation of power spectrum and autocorrelation function as well as the dimension estimation) showed that it is highly periodic and affected by noise. The experimental noise was filtered from it by means of the algorithm described in [Press *et al.*, 1987]. The phase portrait was restored as follows:

$$\mathbf{x} = \left\{ \int_0^t a(\tau)d\tau;\ a(t);\ \frac{da(t)}{dt};\ \frac{d^2a(t)}{dt^2} \right\} \tag{7}$$

and is shown in Fig. 2(b). The fitted dynamical system of the form (1) modeling the given regime possesses the attractor identical to the initial one [Fig. 2(d)] whose realization is shown in Fig. 2(c).

After applying the proposed method for reconstructing a model on a rather simple inhomogeneous time series of biological origin and ensuring its workability, we move to a more complicated example of inhomogeneous realization, namely, the electrocardiogram of a human heart.

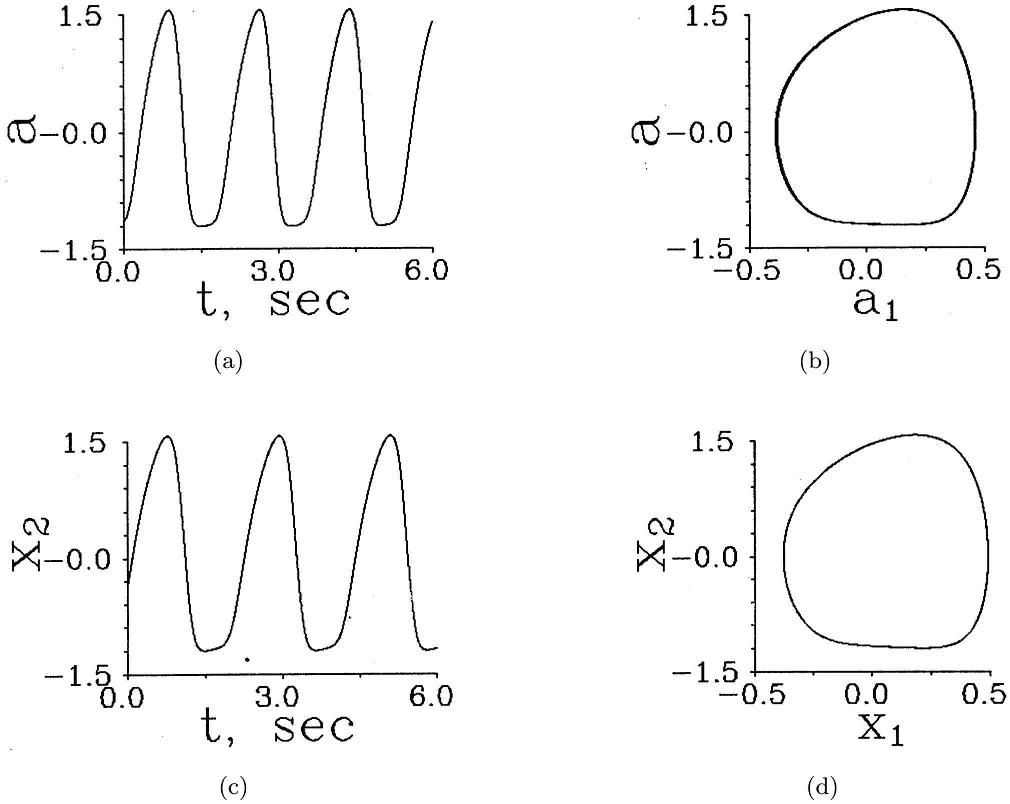


Fig. 2. (a) Initial realization $a(t)$ of mechanical oscillations of a point on the surface of an isolated frog's heart (after noise filtration); (b) projection of phase portrait restored on this realization by the method (4); (c), (d) the realization and the projection of attractor of the corresponding reconstructed DS.

4.2. *Modeling of a human ECG*

Consider an ECG in the frames of the problem of synthesis of DS on an observable and pay attention to the following perculiarities of this signal which do not allow one to directly apply the traditional methods for global reconstruction.

The first perculiarity is the sharp inhomogeneity of a typical ECG which consists of segments with quick motion (containing P and T waves and QRS-complex) and the segments where the motion is abruptly slowed down or absent, i.e. "pauses" [Figs. 4(a) and 5(a)].

The use of successive differentiation method leads to a smooth but sharply inhomogeneous phase portrait [Fig. 3(a)], when the probability of the phase point's staying in the region of "pause" is much larger than the probability of it traveling to the other regions of the attractor.

When applying the delay method we can choose the particular values of time lag τ for which the restored phase portrait is smooth [Fig. 3(b)] but in this case it is also inhomogeneous. This can be simply explained as follows. To obtain a smooth attractor we have to take small values of τ, namely, much less than the duration of "pause", and, therefore, there will be a time interval of finite length (large enough for reasonable embedding dimensions) during which *all* the phase coordinates will not move (or move very slowly). We can also choose such time lags for which the resulting attractor will be more homogeneous in the sense that as one coordinate "stays still" in the period of "pause", at the same time another one moves, being in its "quick phase". But the values of τ required for this will lead to an attractor containing points where it is not smooth [Fig. 3(c)] for which it is rather difficult to fit the phase flow.

Moreover, the minimal embedding dimension required for the reconstruction should be not less than 4 [Babloyantz & Destexhe, 1988], and due to the neccessary presence of noise in experimental observables, the errors of computation of high-order derivatives (second and higher) lead to additional and often unsolvable problems when fitting the right-hand parts of a model.

The second perculiarity of an ECG of a healthy human heart is the fact that its single beat of

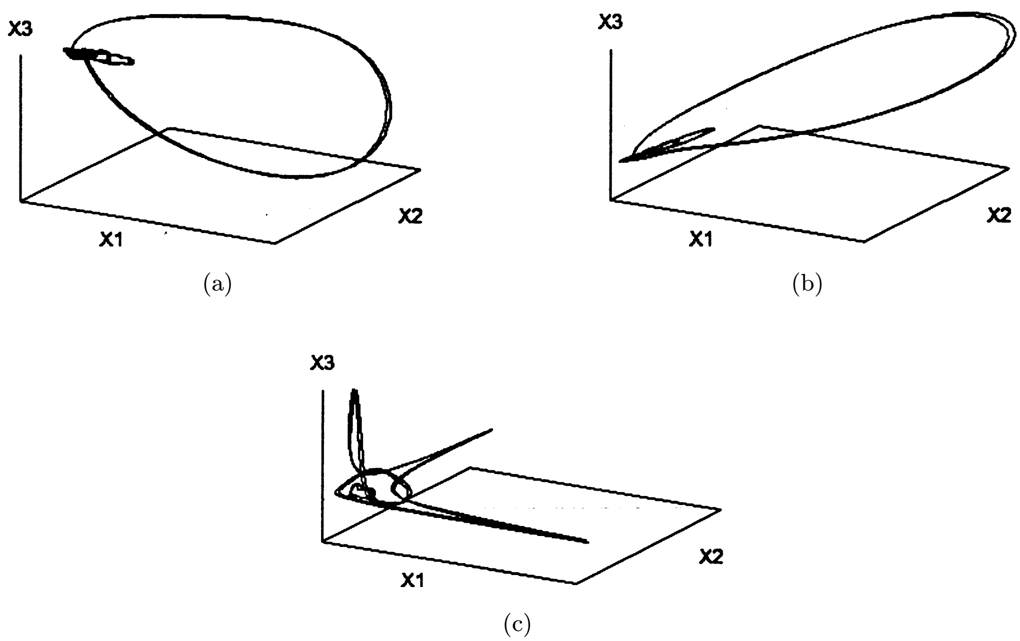


Fig. 3. Phase portrait restored on ECG (a) by means of successive differentiation; (b) by delay method with τ being much less than the duration of "pause"; (c) by delay method with τ being close to the duration of "pause".

duration T_i contains the full information about the structure of PQRST — peaks with a common time duration T_c (with this, the value $T_i - T_c$ is the time duration of "pause").

First, compare visually the two different arbitrarily chosen beats of the same ECG by their simple overlapping. To perform this comparison, the durations of two considered segments were equalized by changing only the duration of "pause" and preserving the PQRST part. Such comparison allows one to speak about the existence of a large similarity between different beats of ECG. To prove this similarity more rigorously we used the coherence function γ_{xy} [Bendat & Piersol, 1989]:

$$\gamma_{xy}^2(f) = \frac{\mid G_{xy}(f) \mid^2}{G_{xx}(f)G_{yy}(f)}\,, \tag{8}$$

where G_{xx}, G_{yy} are the spectral power densities of realizations x and y, respectively, G_{xy} is the mutual spectral power density. As is known, γ_{xy} is equal to 1 if x and y are connected linearly, to 0 if x and y are uncorrelated and to an intermediate value if the connection between them is nonlinear or if the realizations are polluted by noise. However, to correctly apply this characteristic, rather long realizations x and y are required which are absent according to the condition of our problem. To resolve the established contradiction we use the following method: each of the chosen segments of ECG are being repeated many times to obtain a periodic time series of sufficient length. We shall further call this procedure "closing" because it provides the possibility of forming a closed curve of a limit cycle type in the phase space. Since the interval between the R-peaks corresponds to the initial segment under comparison, the closing was performed just with respect to the R-peak due to which the inserted distortions consisted in only an almost negligible variation of the height of this peak.

The coherence function calculated for the resultant time dependencies $x(t)$ and $y(t)$ was averaged over all frequencies to get a certain global characteristic for their comparison. In our case, in the interval 0–40 Hz it equals 1 with an accuracy up to the four decimal points, proving mathematically the similarity of the above segments of ECG.

Taking into account the described perculiarities of ECG we attempt to model a dynamical system whose solution will be a periodic signal reproducing with high accuracy a single period of ECG. One could put such a model in correspondence with an electrocardiogram if the heart beats were periodic, the latter coinciding with the early notion of medical doctors about the functioning of heart in the absence of fluctuations. Since the later investigations disproved this notion [Babloyantz & Destexhe, 1988], such an approach to modeling may be argued. However, one should first answer a question about the possibility of solving such a simplified problem because without this knowledge the

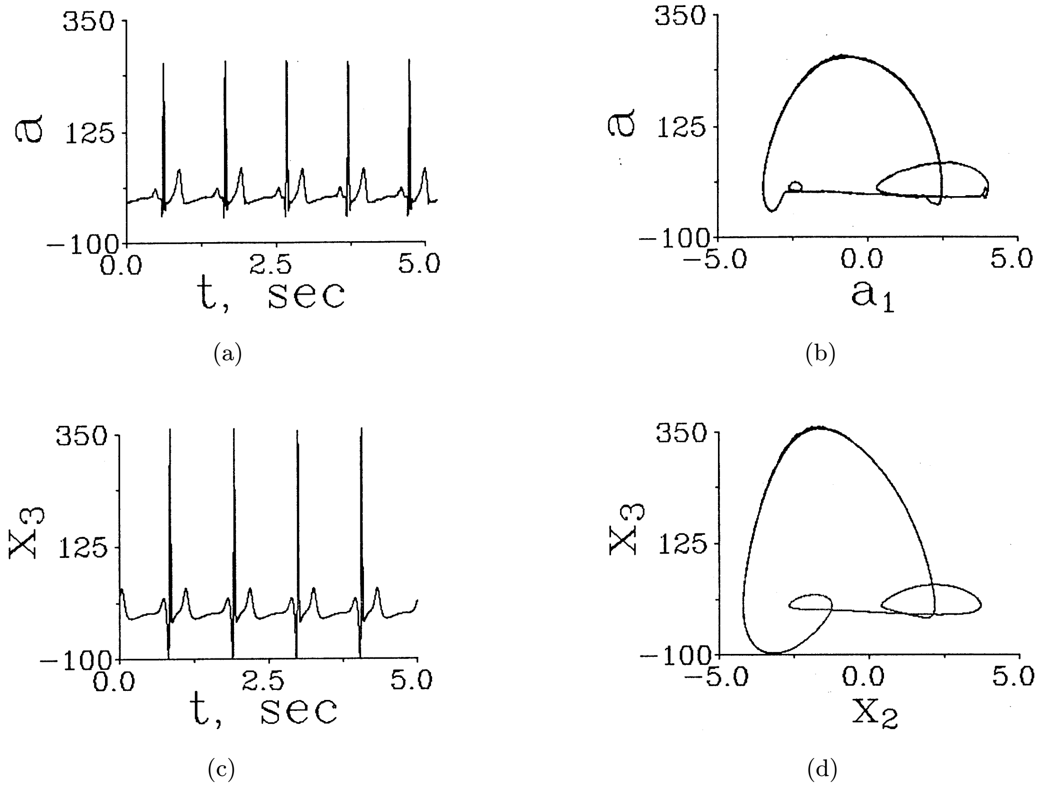


Fig. 4. (a) Initial periodic realization obtained by "closing" a single beat of a real ECG of the first type; (b) projection of phase portrait restored on this realization by the method (4); (c), (d) the realization and the projection of attractor of the corresponding reconstructed DS.

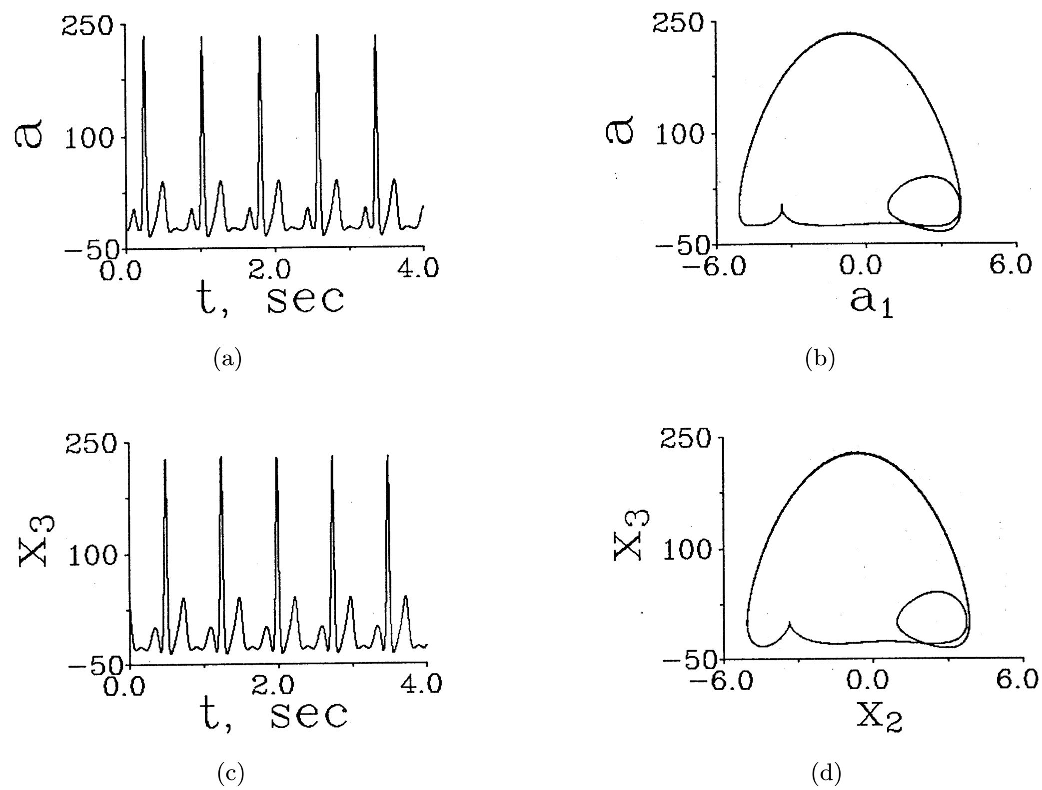


Fig. 5. (a) Initial periodic realization obtained by "closing" a single beat of a real ECG of the second type; (b) projection of phase portrait restored on this realization by the method (4); (c), (d) the realization and the projection of attractor of the corresponding reconstructed DS.

application of reconstruction methods to an ECG loses its meaning.

To solve the task of modeling, two electrocardiograms were chosen which exhibited different types of behavior from which noise was filtered by the method [Press *et al.*, 1987] [Figs. 4(a) and 5(a)]. For each of them a single beat was arbitrarily chosen and subjected to the procedure of closing to obtain sufficiently long realizations.

Since the system under study (human heart) produces the realizations with complicated shape (much more complicated than those considered previously) and the duration of "pause" is rather long compared to the "quick" phase (with sharper inhomogeneity in the latter), it is not enough to use only one integral of the initial coordinate. For this case we successively computed two such integrals to restore the phase vectors as follows. Let $b(t)$ be the initial signal (artificially obtained periodic realization with zero average). We compute

$$a_1(t) = \int_0^t b(\tau)d\tau; \quad a(t) = \int_0^t a_1(\tau)d\tau\,. \qquad (9)$$

Now, consider $a(t)$ as initial realization. By means of successive differentiations the remaining coordinates of phase vector were restored, the latter finally having the following form:

$$\mathbf{x} = \left\{ a(t); \frac{da(t)}{dt}; \frac{d^2a(t)}{dt^2}; \ldots; \frac{d^{N-1}a(t)}{dt^{N-1}} \right\}. \quad (10)$$

It is obvious that d^2a/dt^2 is the initial signal $b(t)$. The phase portrait projection restored by the described way on the chosen beats of two different ECGs are shown in Fig. 4(b) and Fig. 5(b). For these two cases we reconstruct two models (3- and 4-dimensional) in the form of the systems of ODE's Eqs. (1), whose solutions are given in Figs. 4(c), 4(d) and Figs. 5(c), 5(d). We present the explicit form of one of the obtained models together with the particular coefficients of function $f(x_1, x_2, \ldots, x_N)$ in Table 1. We did not succeed in obtaining similar results when using other methods for phase portrait restoration.[1]

Now, let us check the quality of the performed reconstruction by comparing the solutions of the obtained dynamical systems with the initial (for each of them) periodic signals (the corresponding closings of single periods of ECG's) with the help of coherence function. It was computed for the range of frequencies from 0 Hz to 40 Hz, the latter frequency boundary being chosen for the following reasons.

Table 1. The numerical values of the fitted coefficients of the sought function

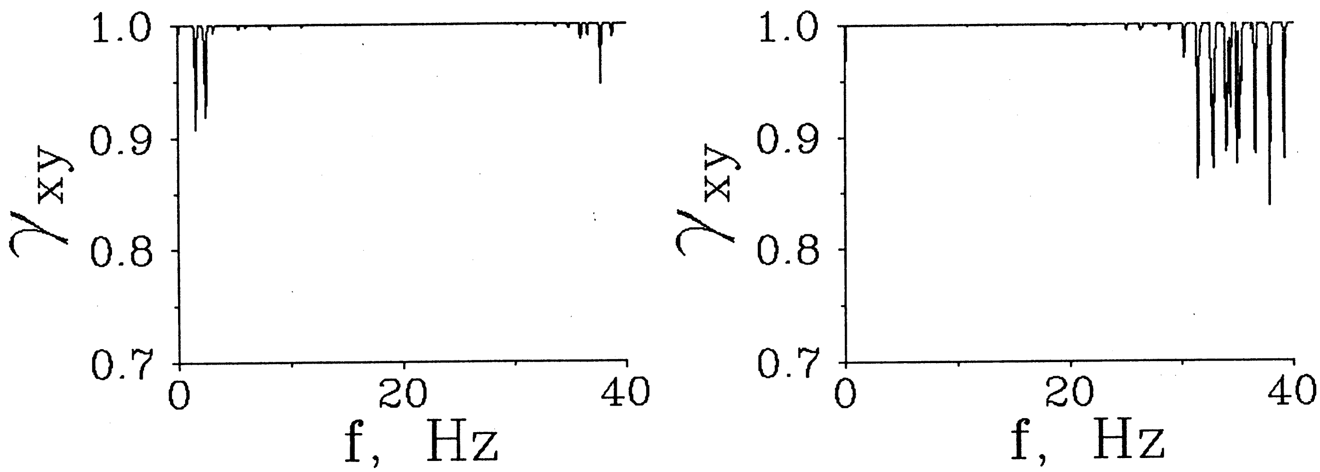
$$f(x_1, x_2, x_3) = \sum_{l_1, l_2, l_3=0}^{3} C_{l_1, l_2, l_3} \cdot x_1^{l_1} x_2^{l_2} x_3^{l_3} \qquad (l_1 + l_2 + l_3 \le 3)$$

in the right-hand part of the global model obtained for one period of ECG. The initial and reconstructed realizations are shown in Figs. 4(a) and 4(c), respectively.

l_1 l_2 l_3	C_{l_1, l_2, l_3}
0 0 0	−560
0 0 1	−180
0 0 2	− 1.31356
0 0 3	− 0.00196
0 1 0	2140
0 1 1	−8.3
0 1 2	0
0 2 0	230
0 2 1	0
0 3 0	−230
1 0 0	−17700
1 0 1	−1240
1 0 2	− 4.4
1 1 0	−3850
1 1 1	90
1 2 0	−70
2 0 0	−46525
2 0 1	−2250
2 1 0	−20000
3 0 0	33618

Filtering the frequencies higher than 40 Hz leads to the smoothing of peaks and almost negligible lessening of their height not significantly distorting the shape of ECG. The above considerations allow us to take the frequency range 0–40 Hz

[1]In paper [Janson & Anishchenko, 1995] we made an attempt to model an ECG with the help of the differentiation method for the phase vector determination. However, the solution of the obtained model system, though containing all the required PQRST-peaks, gave a bad enough local description of the initial signal.



(a) (b)

Fig. 6. Coherence function computed for two pairs "initial signal solution of reconstructed DS" corresponding to two types of behavior of ECG.

as the region of our interest though we should note that the mentioned frequency boarderlines are rather conventional and, besides, depend on a particular ECG.

Computation of coherence function for two pairs of signals give the values for global caracterictic $\approx$ 0.998 and $\approx$ 0.995, respectively. The full plots for the computed dependencies $\gamma_{xy}(f)$ are shown in Figs. 6(a) and 6(b). The given plots as well as the values of global coherence characteristic provide evidence that there exists an almost linear dependence between the solutions of reconstructed model systems and the corresponding initial realizations. And, unlike the results of our previous papers [Janson & Anishchenko, 1995; Anishchenko *et al.*, 1996], the presently obtained models give a good local description of initial signals, testifying to the advantage of the method used.

5. Summary

The results presented allow us to make the following conclusions.

First, the proposed method for phase vector restoration allows one to reconstruct the global model at least for the periodic signals having complicated structure and being sharply inhomogeneous by the traditionally used, simple method of least squares for fitting the sought coefficients. It makes possible the restoration of a sufficiently homogeneous attractor from inhomogeneous data, and thus does not require a more complicated technique to fit the model.

Second, if some of the phase coordinates are restored by the integration method, and the remaining ones by means of differentiation, the resulting system of ODE's have the most simple form [Eqs. (1)].

Third, the discussed integration technique is less sensitive to noise as compared to the derivatives method.

Along with the stated advantages of the method discussed, one should point out a rather severe limitation it has, namely, the requirement of highly stationary experimental realizations. In the present work we illustrated the workability of the method using periodic realizations as examples just to avoid the problem of nonstationarity at the first step. But the mentioned limitation does not mean that it is impossible to apply the integration technique to chaotic data. The only difficulty arising in connection with integration as one deals with chaotic time series is that it is rather difficult to remove the nonstationarity from experimentally measured data. While testing the same method on the numerically computed chaotic realizations of known model systems (Roessler, etc.), we are sure of its workability when it is possible to reach any desired level of stationarity.

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