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Correlation analysis of dynamical chaos

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Abstract

We study correlation and spectral properties of chaotic self-sustained oscillations of different types. It is shown that some classical models of stochastic processes can be used to describe behavior of autocorrelation functions of chaos. The influence of noise on chaotic systems is also considered.

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1. Introduction

The analysis of correlation functions plays an important role in the study of both stochastic and chaotic processes which result from a deterministic dynamics of non-linear systems. The importance of correlation properties is determined by a number of reasons. The presence of mixing causes autocorrelation functions to decay to zero for large times (correlation splitting). This means that the system states separated by a sufficiently large time interval become statistically independent [1–5]. From the property of mixing it follows that a dynamical system is ergodic. Additionally, for chaotic dynamical systems the splitting of correlations in time is connected with an instability of chaotic trajectories and with the system property to produce entropy [1–7].

In spite of their significant importance, correlation properties of chaotic processes have been studied insufficiently. It is widely believed that autocorrelation functions of chaotic systems exponentially decrease at a rate being defined by the Kolmogorov entropy [3]. The Kolmogorov entropy, H_K , in turn is bounded from above by the

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sum of positive Lyapunov exponents [5,7,8]. But this estimation is true only for some special cases.

It has been proven for some classes of discrete maps (expanding and Anosov), which exhibit a mixing invariant measure, that the decay of correlations with time is bounded from above by an exponential function [9–12]. There are different estimations of the rate of this exponential decay which are not always connected with Lyapunov exponents [13–15]. For continuous-time systems, there are no theoretical results at all for estimating the rate of correlation splitting [16].

The studies of specific chaotic systems testify to a complicated behavior of correlation functions, which is defined not only by positive Lyapunov exponents but also by different characteristics and peculiarities of the system chaotic dynamics [13,15,17].

The autocorrelation function (ACF) of stationary ergodic oscillations $x(t)$ is written as follows:

$$\psi(\tau) = \langle x(t)x(t + \tau) \rangle - \langle x \rangle^2, \quad (1)$$

where $\langle x \rangle$ is the mean value of oscillations, and the angular brackets denote averaging over an ensemble. Since the process is ergodic, this procedure can be replaced by time averaging.

In numerical experiments one usually deals with the normalized autocorrelation function $\Psi(\tau) = \psi(\tau)/\psi(0)$.

Consider, for example, a one-dimensional stretching map:

$$x(n + 1) = Kx(n), \quad \text{mod } 1. \quad (2)$$

For $K > 1$, map (2) represents the simplest model of a chaotic system. The property of mixing has been proven for this model [18,19]. By using approximate analytical methods it can be shown that for integer $K \gg 1$, the ACF decays exponentially with a decrement that is equal to $\ln K$ and corresponds to the Kolmogorov entropy. Our numerical calculations indicate that the exponential law $\Psi(m) = \exp(-m \ln |K|)$ holds already for integer $K \geq 2$ (see Fig. 1a). However, for non-integer K and especially when K is very close to 1, the decay of correlations can be significantly different from the exponential behavior (Fig. 1b).

A complicated behavior of autocorrelation functions of chaotic oscillations, that is quite typical for many chaotic systems, depends on numerous factors. Here one can indicate (i) inhomogeneity of the local instability properties in different regions of the phase space, that can lead, as mentioned in Ref. [13], to a slow asymptotics of the ACF, (ii) the existence of almost periodic oscillations, and (iii) the presence of switching-type effects. All the above listed properties are most typical for chaotic attractors of a nonhyperbolic type [20–22], which are realized in different models of real dynamical systems. However, even for nearly hyperbolic attractors (such as, i.e., the Lorenz attractor [20,23,24]) the rate of correlation splitting is defined not only by the rate of an exponential separation of trajectories.

In the present paper we study correlation and spectral properties of chaotic oscillations for several types of chaotic attractors which can be observed in autonomous differential systems with three-dimensional phase space. For our studies we choose

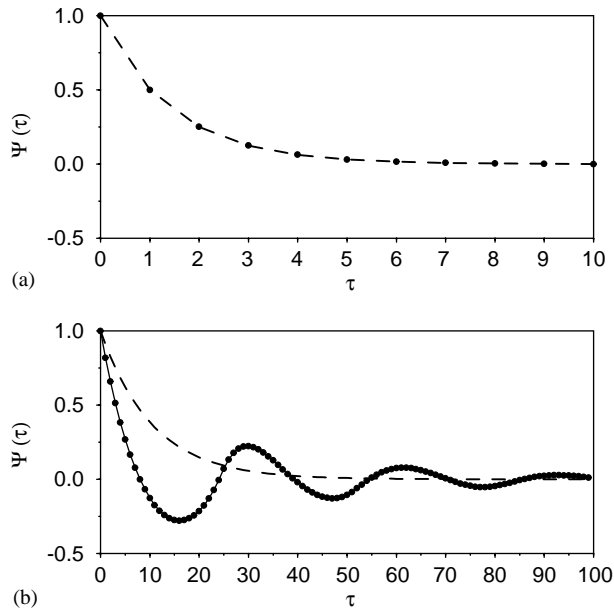


Fig. 1. Normalized ACF of the chaotic sequence $x(n)$ produced by the map (2) for $K=2$ (a) and $K=1.1$ (b). The circles represent the numerical data and the dashed line shows the exponential estimation $\exp(-m \ln|K|)$. In case (a) we add weak noise to the system in order to avoid periodicity that arises for certain integers K due to round-off errors.

classical models of nonlinear dynamics such as the Rössler oscillator [25], the Lorenz system [26], and the Anishchenko–Astakhov oscillator being a mathematical model of a real radiotechnical device [27]. In our paper we made an attempt to answer several fundamental questions. Which peculiarities of the system’s chaotic dynamics can define the rate of correlation splitting and the basic spectral line width? How does noise affect the spectral and correlation characteristics of chaos? Basing on the results of numerical simulation, we would like to show that in the context of correlation properties, different types of chaotic self-sustained oscillations can be associated with basic models of stochastic processes such as harmonic noise and a telegraph signal.

2. Harmonic noise and telegraph signal

When solving applied problems one often has to deal with some models of random processes such as noisy harmonic oscillations and a telegraph signal. The first model is used to describe the influence of natural and technical fluctuations on spectral and correlation characteristics of oscillations of Van der Pol type oscillators [28–30]. The model of telegraph signal serves to outline statistical properties of impulse random processes, for example, random switchings in a bistable system in the presence of noise (the Kramers problem, noise-induced switchings in the Schmitt trigger, etc.

[30–32]). Experience of the studies of chaotic oscillations in three-dimensional differential systems shows that the aforementioned models of random processes can be used to describe spectral and correlation properties of a certain class of chaotic systems. As we will demonstrate below, the model of harmonic noise represents sufficiently well correlation characteristics of spiral chaos, while the model of telegraph signal is quite suitable for studying statistical properties of attractors of the switching type, such as attractors in the Lorenz system [26] and in the Chua circuit [33].

In the following we summarize the main characteristics of the above mentioned classical models of random processes.

Harmonic noise is a stationary random process with zero mean. It is represented as follows [28–30]:

$$x(t) = R_0[1 + \alpha(t)] \cos[\omega_0 t + \phi(t)], \quad (3)$$

where R_0 and ω_0 are constant (average) values of the amplitude and frequency of oscillations, respectively; $\alpha(t)$ and $\phi(t)$ are random functions that characterize amplitude and phase fluctuations, respectively. The process $\alpha(t)$ is assumed to be stationary.

Several simplifying assumptions which are most often used are as follows: (i) the amplitude and phase fluctuations are statistically independent, and (ii) the phase fluctuations $\phi(t)$ represent a Wiener process with a diffusion coefficient B . Under the assumptions made, the ACF of the process (3) can be written as follows [28–30]:

$$\psi(t) = \frac{R_0^2}{2} [1 + K_\alpha(\tau)] \exp(-B|\tau|) \cos \omega_0 \tau, \quad (4)$$

where $K_\alpha(\tau)$ is the covariation function of reduced amplitude functions $\alpha(t)$.¹ Using the Wiener–Khinchin theorem one can derive the corresponding expressions for the spectral power density.

Generalized telegraph signal. This process describes random switchings between two possible states $x(t) = \pm a$. Two main kinds of telegraph signals are usually considered, namely, random and quasi-random telegraph signals [30,34]. A random telegraph signal is characterized by a Poissonian distribution of switching moments t_k . The latter leads to the fact that the impulse duration θ has the exponential distribution:

$$\rho(\theta) = n_1 \exp(-n_1 \theta), \quad \theta \geq 0, \quad (5)$$

where n_1 is the mean switching frequency. The ACF of such a process can be represented as follows:

$$\psi(\tau) = a^2 \exp(-2n_1|\tau|). \quad (6)$$

Another type of telegraph signal (a quasi-random telegraph signal) corresponds to random switchings between the two states $x(t) = \pm a$, which can occur only in discrete time moments $t_n = n\xi_0 + \alpha$, $n = 1, 2, 3, \dots$, where $\xi_0 = \text{const}$ and α is a random quantity. If the probability of switching events is equal to $1/2$, then the ACF of this process is

¹ Prefactor $R_0^2[1 + K_\alpha(\tau)]$ is the covariation function $K_A(\tau)$ of the random amplitude $A(t) = R_0[1 + \alpha(t)]$. This notion is most convenient to use in our further studies.

given by the following expression:

$$\begin{aligned} \psi(\tau) &= a^2 \left(1 - \frac{|\tau|}{\xi_0} \right) \quad \text{if } |\tau| < \xi_0 ; \\ \psi(\tau) &= 0 \quad \text{if } |\tau| \geq \xi_0 . \end{aligned} \tag{7}$$

3. Correlation and spectral analysis of spiral chaos

Spiral (or phase-coherent) attractors are formed through a sequence of period-doubling bifurcations and are referred to as chaotic attractors of nonhyperbolic type [27,35,36]. The power spectrum of spiral chaos exhibits a well-pronounced peak at the basic (average) frequency and, consequently, the envelope of the ACF decays relatively slowly. Spiral attractors can be observed in the Rössler system [25], in the Anishchenko–Astakhov oscillator [27], in the Chua circuit [33], etc. Self-sustained oscillations in these systems resemble dynamics of noisy periodic oscillators. In particular, they are characterized by a finite width of the spectral line at the basic frequency and demonstrate effects of forced and mutual synchronization [37,38]. From a physical viewpoint, chaotic attractors of the spiral type possess the properties of a noisy limit cycle. However, spiral attractors are realized in fully deterministic systems, i.e., without external fluctuations.

Consider the regime of spiral chaos in the Rössler system:

$$\begin{aligned} \dot{x} &= -y - z + \sqrt{2D}\zeta(t) , \\ \dot{y} &= x + 0.2y , \\ \dot{z} &= 0.2 - z(\mu + x), \quad \mu = 6.5 , \end{aligned} \tag{8}$$

where $\zeta(t)$ is the normalized Gaussian source of delta-correlated noise with zero mean, and D is the noise intensity. Let us introduce the substitution of variables

$$\begin{aligned} x(t) &= A(t) \cos \Phi(t) , \\ y(t) &= A(t) \sin \Phi(t) , \end{aligned} \tag{9}$$

which defines the amplitude $A(t)$ and the full phase $\Phi(t)$ of the chaotic oscillations.

In Ref. [39] it has been recently shown that for spiral chaos in the Rössler system the variance of the instantaneous phase grows linearly in time both without noise ($D = 0$) and when $D \neq 0$. The variance of the total phase is equal to the variance of its non-regular component $\phi(t) = \Phi(t) - \omega_0 t$, where $\omega_0 = \langle \dot{\Phi}(t) \rangle$ is the mean frequency of the chaotic oscillations. The linear dependence of the variance $\sigma_\phi^2(t)$ on time allows us to introduce the effective phase diffusion coefficient

$$B_{\text{eff}} = \frac{1}{2} \frac{d\sigma_\phi^2(t)}{dt} . \tag{10}$$

In our numerical simulation of Eqs. (8) we calculate the normalized autocorrelation function of the chaotic oscillations $x(t)$. Using Eqs. (9) we compute the covariance

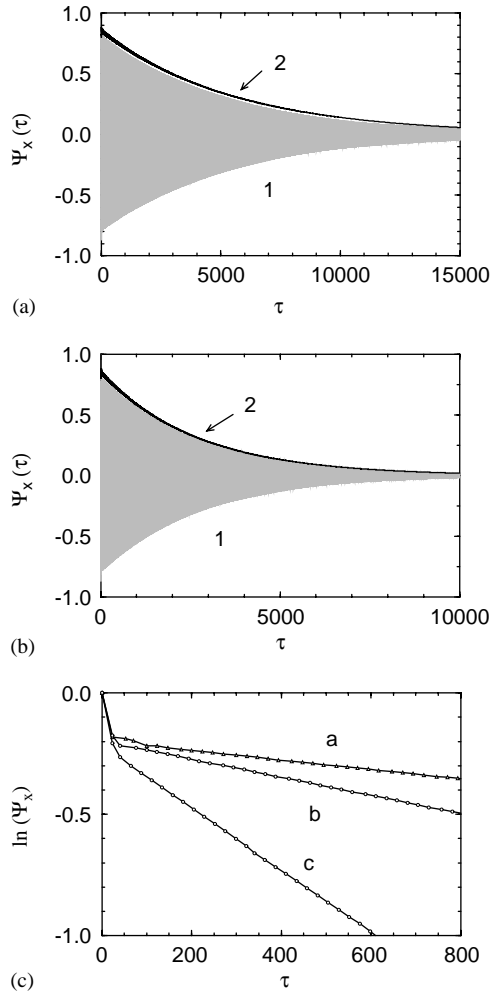


Fig. 2. Normalized ACF of the $x(t)$ oscillations in system (8) for $\mu=6.5$ (grey dots 1) and its approximation by (11) (black dots 2) for $D=0$ (a) and $D=10^{-3}$ (b). (c) The envelopes of ACF in a linear-logarithmic scale for $D=0$ (curve a), $D=0.001$ (curve b), and $D=0.01$ (curve c).

function of the amplitude $K_A(\tau)$ and the effective phase diffusion coefficient B_{eff} . We use the time-averaging procedure for calculating $\Psi_x(\tau)$ and $K_A(\tau)$. The coefficient B_{eff} is computed by averaging over an ensemble of realizations [39]. Fig. 2 shows the calculation results for $\Psi_x(\tau)$ in system (8). The ACF decays almost exponentially both without noise (Fig. 2a) and in the presence of noise (Fig. 2b). Additionally, as seen from Fig. 2c for $\tau < 20$ there is an interval on which the correlations decrease much faster.

Using Eq. (4) we can approximate the envelope of the calculated ACF $\Psi_x(\tau)$. To do this, we substitute the numerically computed characteristics $K_A(\tau)$ and $B=B_{\text{eff}}$ into

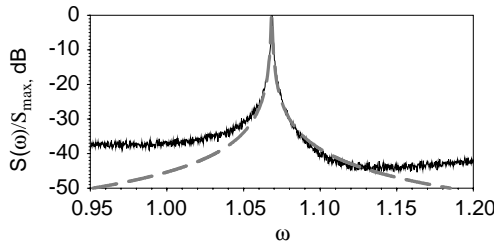


Fig. 3. A part of the normalized power spectrum of $x(t)$ oscillations in system (8) for $\alpha = \beta = 0.2$, and $\mu = 6.5$ (solid line) and its approximation by Eq. (12) (dashed line) for the noise intensity $D = 10^{-3}$.

an expression for the normalized envelope $\Gamma(\tau)$:

$$\Gamma(\tau) = \frac{K_A(\tau)}{K_A(0)} \exp(-B_{\text{eff}}|\tau|). \tag{11}$$

The calculation results for $\Gamma(\tau)$ are shown in Fig. 2a,b by black dots (curves 2). It is seen that the behavior of the envelope of $\Psi_x(\tau)$ is described well by Eq. (11). Note that taking into account the multiplier $K_A(\tau)/K_A(0)$ enables us to obtain a good approximation for all times τ . This means that the amplitude fluctuations play a significant role on short time intervals, while the slow process of the correlation decay is mainly determined by the phase diffusion. Thus, we can observe a surprisingly good agreement between the numerical results for the spiral chaos and the data for the classical model of harmonic noise. At the same time, it is quite difficult to explain rigorously the reason of such a good agreement. Firstly, the relationship (4) was obtained by assuming the amplitude and phase values to be statistically independent. However, this approach cannot be applied to a chaotic regime. Secondly, when deriving (4) we used the fact that the phase fluctuations are described by a Wiener process. In the case of chaotic oscillations, $\phi(t)$ is a more complicated process and its statistical properties are unknown. It is especially important to note that the findings presented in Fig. 2a were obtained in the regime of purely deterministic chaos, i.e., without noise in the system.

We have shown that for $\tau > \tau_{\text{cor}}$ the envelope of the ACF for the chaotic oscillations can be approximated by the exponential law $\exp(-B_{\text{eff}}|\tau|)$. Then according to the Wiener–Khinchin theorem, the spectral peak at the average frequency ω_0 must have a Lorentzian shape and its width is defined by the effective phase diffusion coefficient B_{eff} :

$$S(\omega) = C \frac{B_{\text{eff}}}{B_{\text{eff}}^2 + (\omega - \omega_0)^2}, \tag{12}$$

$C = \text{const}.$

The calculation results presented in Fig. 3 justify this statement. The basic spectral peak is approximated by using Eq. (12) and this fits quite well with the numerical results for the power spectrum of the $x(t)$ oscillations. We note that the findings shown in Figs. 2 and 3 for the noise intensity $D = 10^{-3}$ have also been verified for different

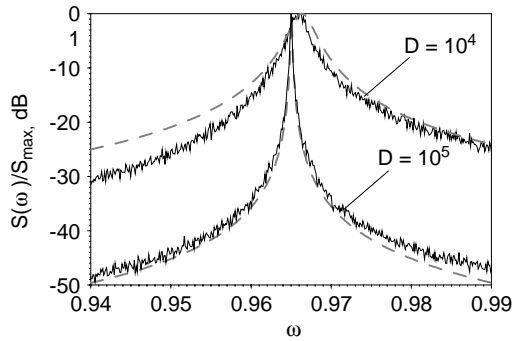


Fig. 4. A part of the normalized power spectra for the Anishchenko–Astakhov oscillator (solid lines) and their approximations by Eq. (12) (dashed lines) for two different noise intensities $D = 10^{-4}$ and 10^{-5} .

values of D , $0 < D < 10^{-2}$, as well as for the range of parameter μ values which correspond to the regime of spiral chaos.

Our findings for the approximation of the ACF and the shape of the basic spectral peak are completely confirmed by our investigations of spiral attractors in other dynamical systems. We exemplify this by considering the Anishchenko–Astakhov oscillator [27]. Fig. 4 demonstrates the parts of the power spectra in the neighborhood of the basic frequency and the corresponding approximations by Eq. (12) for two different noise intensities.

4. Autocorrelation functions and power spectra for funnel chaos

The results obtained for spiral chaos and presented in the previous section can also be generalized to some extent to the regime of funnel chaos. Compared to spiral chaos, a chaotic attractor of the funnel type is characterized by a more complicated rotation of trajectories about an equilibrium point, which is given by a deterministic evolution operator. The rotating behavior is accompanied by disruptions of instantaneous phase values, that can lead to a non-monotonic phase dependence on time [36,40,39]. The funnel chaos can be exemplified by the chaotic regime that is realized in the system (8) for $\alpha = \beta = 0.2$ and $\mu > 8.5$.

We use the concept of analytical signal [41–43] to introduce rigorously the instantaneous phase of $x(t)$ oscillations in the regime with complex phase dynamics. The analytical signal $w(t)$ is a complex function of time that can be defined as follows:

$$w(t) = x(t) + i\tilde{x}(t) = a(t) \exp[i\Phi(t)] , \tag{13}$$

where $\tilde{x}(t)$ is the Hilbert transform of the initial process $x(t)$. The instantaneous phase $\Phi(t)$ of $x(t)$ reads

$$\Phi(t) = \arctan\left(\frac{\tilde{x}}{x}\right) + \pi k , \tag{14}$$

$$k = 0, 1, 2, 3, \dots .$$

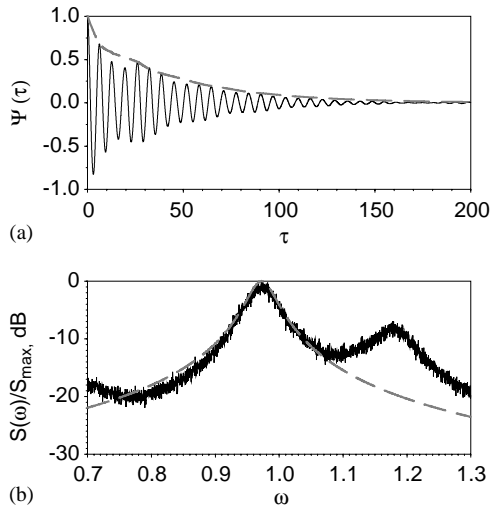


Fig. 5. Normalized ACF (a) and a part of the power spectrum (b) for the funnel chaos in the system (8) with $\mu=13$ and $D=0$. The dashed lines correspond to approximations (11) and (12). The diffusion coefficient is $B_{\text{eff}} = 0.02186$.

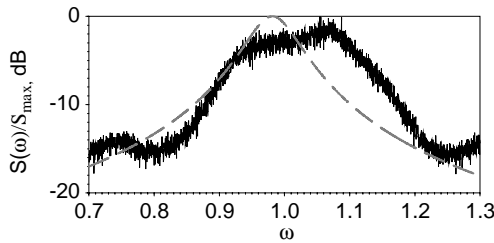


Fig. 6. A part of the power spectrum for the funnel chaos in the system (8) with $\mu=10$ and $D=0$. The dashed lines show the approximation (12). The diffusion coefficient is $B_{\text{eff}} = 0.04009$.

When passing to the funnel attractor, the diffusion coefficient B_{eff} of deterministic chaos increases dramatically (by 2 or 3 orders). This causes the ACF to decay more rapidly and the basic spectral peak to expand significantly [40,39]. The numerical results obtained for system (8) with $\mu=13$ and $D=0$ are shown in Fig. 5. They demonstrate that in the regime of funnel chaos the considered approximations can reproduce quite well the numerical findings.

However, for certain parameter μ values the phase variable $\Phi(t)$ shows a so complicated behavior that the approximation (11) is no longer valid and the basic spectral peak does not even approximately bear a resemblance to a Lorentzian, see Fig. 6.

5. Correlation characteristics of the Lorenz attractor

In Sections 3 and 4 we have used the effective phase diffusion coefficient to describe the correlation properties of the Rössler system and the Anishchenko–Astakhov oscillator. However, such an approach cannot be applied to approximate autocorrelation functions of chaotic oscillations of a switching type. Some chaotic attractors demonstrating a rather complex structure can contain certain regions which are separated by manifolds of saddle points and cycles. Transitions (switchings) between these regions can occur provided that certain conditions are fulfilled [44]. Such oscillations can be observed, for example, in the Lorenz system:

$$\begin{aligned} \dot{x} &= -10(x - y), \\ \dot{y} &= -xz - y + 28x, \\ \dot{z} &= -\frac{8}{3}z + xy. \end{aligned} \tag{15}$$

In the phase space of the Lorenz system there are two saddle-foci that are symmetrical about the z -axis and are separated by the stable manifold of a saddle point in the origin. This stable manifold has a complex structure that allows the trajectories to switch between the saddle-foci in specific paths [20,44] (see Fig. 7). Unwinding about one of the saddle-foci the trajectory approaches the stable manifold and then can jump to the other saddle-focus with a certain probability. The rotation about the saddle-foci does not contribute considerably to the decay of the ACF, while the frequency of “random” switchings essentially affects the rate of the ACF decay. Consider the time series of the x coordinate of the Lorenz system, that is shown in Fig. 8. If one introduces a symbolic dynamics, i.e., one excludes the rotation about the saddle-foci, one obtains a telegraph-like signal. Fig. 9 shows the ACF of the x oscillations for the Lorenz attractor and the ACF of the corresponding telegraph signal. Comparing these two figures we can state that the time of the correlation decay and the behavior of the ACF on this

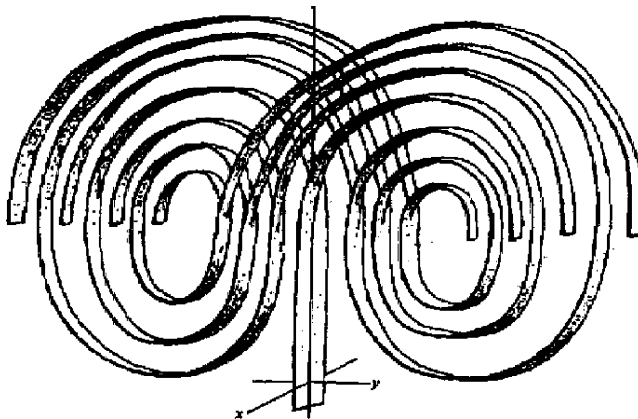


Fig. 7. Qualitative illustration of the structure of manifolds in the Lorenz system.

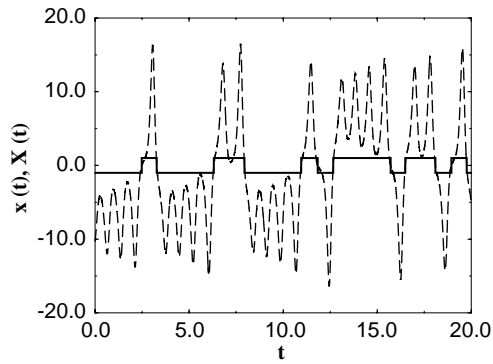


Fig. 8. Telegraph signal (solid curve) obtained for the $x(t)$ oscillations (dashed curve) of the Lorenz system.

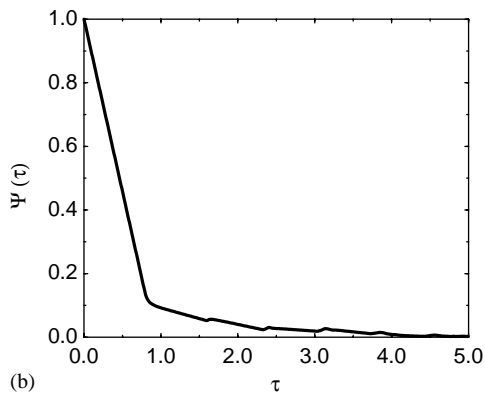
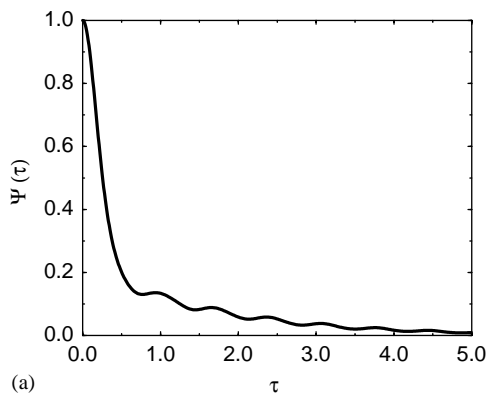


Fig. 9. The ACF of the $x(t)$ oscillations (a) and of the telegraph signal (b).

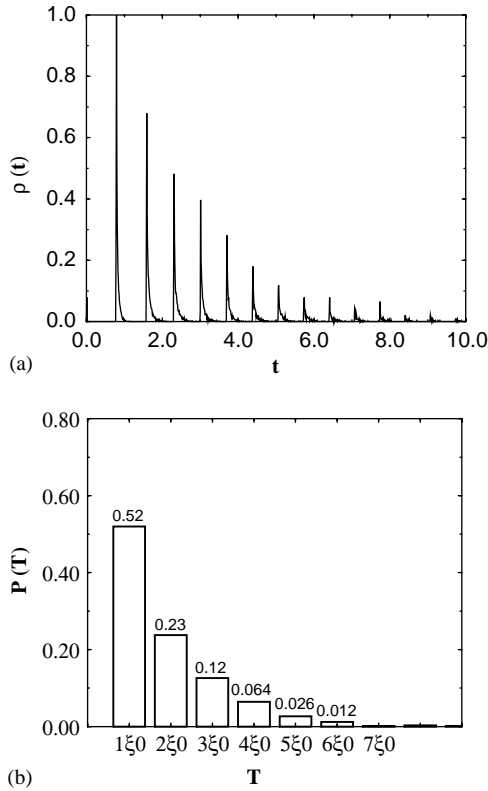


Fig. 10. The distribution of impulse durations of the telegraph signal (a) and probabilities of transitions at times multiple to ξ_0 (b).

time scale are predominantly determined by switchings, whereas the rotation about the saddle-foci makes a minor contribution to the ACF decay on large times. It is worth noting that the ACF decreases linearly on short times. This fact is remarkable as the linear decaying of the ACF corresponds to a discrete equidistant residence time probability distribution in the form of delta-peaks. Additionally, the probability of switchings between the two states is equal to 1/2 [30,34].

Fig. 10 shows the residence time distribution calculated for the telegraph signal resulting from switchings in the Lorenz system. As can be seen from Fig. 10(a), the residence time distribution in the two attractor regions really has a structure that is quite similar to an equidistant discrete distribution. At the same time the peaks are characterized by a finite width. Fig. 10(b) represents the probability distribution of switchings which occur at multiples of ξ_0 , where ξ_0 is the minimal residence time in one of the states. This dependence shows that the probability of transition at time ξ_0 is close to 1/2. The discrete character of switchings can be explained by peculiarities of the structure of the manifolds of the Lorenz system (see Fig. 7). In the vicinity of the origin $x = 0, y = 0$ the manifolds split into two leaves. This leads to the fact that

the probability of switchings between the two states in one revolution about the fixed point is approximately equal to $1/2$. This particular aspect of the dynamics ensures that the ACF of the $x(t)$ and $y(t)$ oscillations on the Lorenz attractor has the form defined by expression (7). However, the finite width of the peaks in the distribution and deviations from the probability $1/2$ can lead to an ACF that decays to a certain finite, nonvanishing value.

6. Conclusion

We have shown in our numerical simulation that the spiral chaos retains to a great extent the spectral and correlation properties of quasiharmonic oscillations. With this, the rate of correlation splitting in a differential system depends on short times on both the instantaneous amplitude and the instantaneous phase diffusion. The width of the basic peak in the power spectrum of the spiral chaos is correspondingly defined by B_{eff} and oscillations of the instantaneous amplitude determine the level of the spectrum background. The effective phase diffusion coefficient in a noise-free system is defined by its chaotic dynamics but is not directly related to the positive Lyapunov exponent.

Our studies of statistical properties of the Lorenz attractor have demonstrated that the properties of the ACF is mainly defined by a random switching process and slightly depends on the rotation about the saddle-foci. The classical model of telegraph signal enables one to describe the behavior of $\psi(\tau)$ for the Lorenz attractor by using expression (7). In particular, this expression approximates quite well a linear decay of the ACF from 1.0 to 0.2 that allows to estimate theoretically the correlation time. The power spectrum of the Lorenz attractor both in a flow and in the Poincaré map was studied in Ref. [17] by applying the symbolic dynamics methods. Already in this paper it has been established that the power spectrum is not a Lorenzian. Our results obtained by using the model of telegraph signal are in a good agreement with the findings presented in Ref. [17].

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References

- [1] P. Billingsley, *Ergodic Theory and Information*, Wiley, New York, 1965.
- [2] I.P. Cornfeld, S.V. Fomin, Ya. Sinai, *Ergodic Theory*, Springer, New York, 1982.
- [3] G.M. Zaslavsky, *Chaos in Dynamical Systems*, Harwood Academic Publishers, New York, 1985.
- [4] Ya. Sinai, Stochasticity of dynamical systems, in: A.V. Gaponov-Grekhov (Ed.), *Nonlinear waves*, Nauka, Moscow, 1979, pp. 192–212 (in Russian).
- [5] J.P. Eckmann, D. Ruelle, *Rev. Mod. Phys.* 57 (1985) 617.
- [6] A.N. Kolmogorov, *Dokl. Acad. Nauk USSR* 124 (4) (1959) 754 (in Russian).

- [7] Ya. Sinai, *Dokl. Acad. Nauk USSR* 124 (4) (1959) 768 (in Russian).
- [8] Ya. Pesin, *Russ. Math. Surveys* 32 (4) (1977) 55.
- [9] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*. Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, 1975.
- [10] D. Ruelle, *Am. J. Math.* 98 (1976) 619.
- [11] M.L. Blank, *Stability and Localization in Chaotic Dynamics*, Moscow Center for Cont. Math. Educ., Moscow, 2001 (in Russian).
- [12] D. Ruelle, *Math. Phys.* 125 (1989) 239.
- [13] F. Christiansen, G. Paladin, H.H. Rugh, *Phys. Rev. Lett.* 65 (1990) 2087.
- [14] C. Liverani, *Ann. Math.* 142 (1995) 239.
- [15] G. Froyland, *Comm. Math. Phys.* 189 (1997) 237.
- [16] R. Bowen, D. Ruelle, *Invent. Math.* 29 (1975) 181.
- [17] R. Badii, M. Finardi, G. Broggi, M.A. Sepúlveda, *Physica D* 58 (1992) 304.
- [18] A. Renyi, *Acta Math. Acad. Sci. Hungary* 8 (1957) 477.
- [19] V.A. Rokhlin, *Izv. Acad. Nauk USSR* 25 (4) (1961) 499.
- [20] L.P. Shilnikov, *Int. J. Bifurc. Chaos* 7 (1997) 1953.
- [21] V.S. Afraimovich, *Attractors*, in: A.V. Gaponov, M.I. Rabinovich, J. Engelbrechet (Eds.), *Nonlinear Waves—I*, Springer, Berlin, Heidelberg, 1989, pp. 14–28.
- [22] V.S. Anishchenko, V.V. Astakhov, A.B. Neiman, T.E. Vadivasova, L. Schimansky-Geier, *Nonlinear Dynamics of Chaotic and Stochastic Systems*, Springer, Berlin, Heidelberg, 2002.
- [23] V.S. Afraimovich, V.V. Bykov, L.P. Shilnikov, *Dokl. Acad. Nauk* 234 (1977) 336 (in Russian).
- [24] V.V. Bykov, L.P. Shilnikov, in: L.P. Shilnikov (Ed.), *Methods of Qualitative Theory and the Theory of Bifurcations*, Gorky State University, Gorky, 1989, pp. 151–159.
- [25] O.E. Rössler, *Phys. Lett. A* 57 (1976) 397.
- [26] E.N. Lorenz, *J. Atmos. Sci.* 20 (1963) 130.
- [27] V.S. Anishchenko, *Complex Oscillations in Simple Systems*, Nauka, Moscow, 1990 (in Russian).
- [28] R.L. Stratonovich, *Selected Problems of the Theory of Fluctuations in Radiotechnics*, Soviet Radio, Moscow, 1961 (in Russian).
- [29] A.N. Malakhov, *Fluctuations in Auto-Oscillating Systems*, Nauka, Moscow, 1968 (in Russian).
- [30] S.M. Rytov, *Introduction in Statistical Radiophysics*, Nauka, Moscow, 1966 (in Russian).
- [31] H.A. Kramers, *Physica* 7 (1940) 284.
- [32] P. Hänggi, P. Talkner, M. Borkovec, *Rev. Mod. Physics* 62 (1990) 251.
- [33] R.N. Madan (ed.), *Chua's circuit: A paradigm for chaos*, World Scientific, Singapore, 1993.
- [34] V.I. Tikhonov, M.A. Mironov, *Markovian Processes*, Soviet Radio, Moscow, 1977 (in Russian).
- [35] J. Crutchfield, et al., *Phys. Lett. A* 76 (1980) 1.
- [36] A. Arneodo, P. Collet, C. Tresser, *Commun. Math. Phys.* 79 (1981) 573.
- [37] V.S. Anishchenko, T.E. Vadivasova, D.E. Postnov, M.A. Safonova, *Int. J. Bifurc. Chaos* 2 (1992) 633.
- [38] M. Rosenblum, A. Pikovsky, J. Kurths, *Phys. Rev. Lett.* 76 (1996) 1804.
- [39] V.S. Anishchenko, T.E. Vadivasova, A.S. Kopeikin, J. Kurths, G.I. Strelkova, *Phys. Rev. E* 65 (2002) 036206.
- [40] V.S. Anishchenko, T.E. Vadivasova, A.S. Kopeikin, J. Kurths, G.I. Strelkova, *Phys. Rev. Lett.* 87 (2001) 4101.
- [41] D. Gabor, *J. Inst. Electr. Eng. (London)* 93 (1946) 429.
- [42] L.A. Vanshtein, D.E. Vakman, *Frequency Separation in the Theory of Oscillations and Waves*, Nauka, Moscow, 1983.
- [43] D. Bendat, A. Pirsol, *Applied Analysis of Random Quantities*, Mir, Moscow, 1989.
- [44] E.A. Jackson, *Perspectives of Nonlinear Dynamics*, Vol. 1, 2, Cambridge University Press, Cambridge, 1989, 1991.