Autocorrelation function and spectral linewidth of spiral chaos in a physical experiment

Vadim S. Anishchenko,[1] Tatjana E. Vadivasova,[1] Jürgen Kurths,[2] George A. Okrokvertskhov,[1] and Galina I. Strelkova[1]
[1]*Department of Physics, Institute of Nonlinear Dynamics, Saratov State University, 410012 Saratov, Russia*
[2]*Group of Nonlinear Dynamics, Institute of Physics, University of Potsdam, D-14415 Potsdam, Germany*
(Received 29 October 2003; published 31 March 2004)

We present results of physical experiments where we measure the autocorrelation function (ACF) and the spectral linewidth of the basic frequency of a spiral chaotic attractor in a generator with inertial nonlinearity both without and in the presence of external noise. It is shown that the ACF of spiral attractors decays according to an exponential law with a decrement which is defined by the phase diffusion coefficient. It is also established that the evolution of the instantaneous phase can be approximated by a Wiener random process.

DOI: 10.1103/PhysRevE.69.036215 PACS number(s): 05.45.Pq, 05.40.−a

Chaotic attractors can essentially differ in their structure and properties. The basic theoretical results were obtained only for robust hyperbolic attractors that represent an ideal model of deterministic chaos. In numerical and physical experiments we deal, as a rule, with nonhyperbolic chaotic attractors for which there is still no exhausting theoretical description. The decay of correlations is one of the fundamental properties of chaotic attractors in the absence of noise [1–4]. The presence of mixing was rigorously proven for a certain class of discrete-time hyperbolic chaotic systems. With this, the decrease of autocorrelation functions can be generally estimated above by an exponential function [1,2]. However, at the present time there are no rigorous theoretical results regarding the behavior of autocorrelation functions in flow chaotic systems with nonhyperbolic attractors.

A phase coherent or spiral attractor is one of the typical nonhyperbolic chaotic regimes [5,6] that can be realized in well-known systems such as the Rössler system, the Belousov-Zhabotinsky reaction model, the Anishchenko-Astakhov system, the Chua system, and many others [6–8]. Trajectories on a spiral attractor rotate about a saddle focus almost periodically and thus, the power spectrum exhibits a narrow peak at the frequency ω_0 that coincides with the average rotation frequency. The spiral chaos can be described in terms of the instantaneous amplitude and phase [9–11]. In our recent works [12–15] we have shown numerically that the rate of ACF decay and the fundamental spectral linewidth in the regime of phase-coherent chaos can be mainly defined by the effective diffusion coefficient of the instantaneous phase of oscillations. Statistical characteristics of a deterministic chaotic system appear to be similar to the properties of oscillations of a noisy Van der Pol oscillator [14].

The objective of this paper is to answer the following fundamental question: Whether the regularities established numerically will be fulfilled in physical experiments with a generator of spiral chaos? The answer to this question is far from evident and is principally related to the correspondence between models and real physical systems. Therefore, the performance of a physical experiment is of high importance.

In Refs. [12–16], stochastic equations of the Rössler model and of the Anishchenko-Astakhov generator were studied in the presence of additive white noise. It is absolutely obvious that a physical nonlinear dissipative system must be described by stochastic equations that include both additive and multiplicative noise sources. Unfortunately, the formulation of adequate stochastic equations for the indicated systems involves insurmountable difficulties. One of the most effective methods to overcome them is to perform physical experiments whose results are presented in this paper. The objective of such experiments is to measure the auto correlation function (ACF) and the spectral linewidth of spiral chaos and to compare experimental findings with the theoretical and numerical results described in Refs. [12,14–16]. The experiments were conducted on an experimental unit that consisted of a radio-technical generator with inertial nonlinearity (the Anishchenko-Astakhov generator, GIN) [6] having the basic frequency 18,5 kHz, a computer with a fast analog-to-digital converter (ADC) with the discretization frequency 694,5 kHz, and a Gaussian broadband noise generator with a frequency range from 0 kHz to 100 kHz. Block schemes of the GIN and the experimental unit are shown in Fig. 1. The behavior of the ACF was also analyzed in the presence of noise. With this purpose, a broadband noise from the external noise generator was applied to the system, and the noise intensity could be varied. The generator with inertial nonlinearity is described by a simple three-dimensional dynamical system, which is as follows:

$$\dot{x}=mx+y-xz-\delta x^3, \quad \dot{y}=-x,$$

$$\dot{z}=-gz+gI(x)x^2, \quad I(x)=\begin{cases}1, & x>0\\ 0, & x\leqslant 0.\end{cases} \qquad (1)$$

The system (1) can demonstrate the regimes of spiral chaos for certain values of the parameters m and g [6].

The first important question to be uniquely answered by the experiment is whether a Wiener process approach can be applied to describe the phase of the $x(t)$ process, as assumed in Refs. [12,15,16,9]. To define the diffusion coefficient B_{eff}, the instantaneous phase is introduced by using an analytical signal concept and performing the Hilbert transform for experimental realizations $x(t)$ [11]. Then the phase variance $\sigma_\phi^2(t)$ is calculated by averaging over an ensemble of N realizations. The effective phase diffusion coefficient is defined by the rate of the variance growth in time. The temporal dependence of the phase diffusion shown in Fig. 2 is not rigorously linear as it must be observed for the Wiener pro-

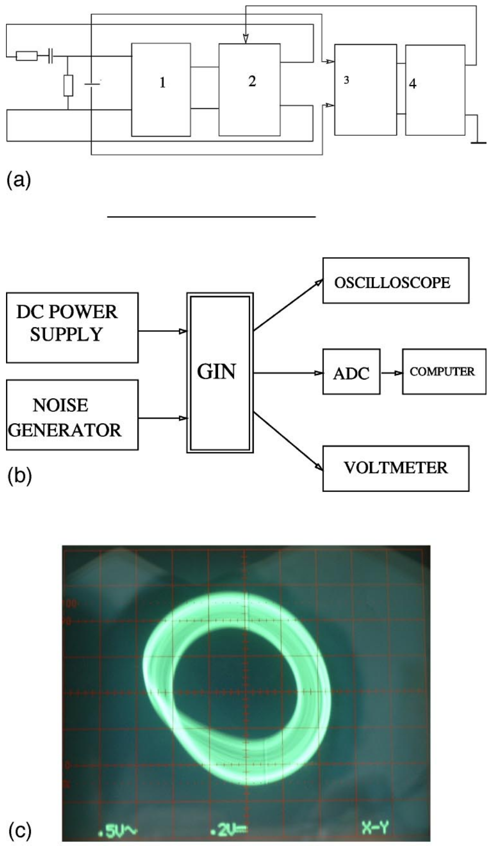

FIG. 1. Schemes of a generator with inertial nonlinearity (a) (1,2—amplifiers, 3—linear amplifier, 4—inertial convertor) and of an experimental unit (b), and a picture of the spiral attractor on the oscilloscope screen (c).

cess. However, the linear growth dominates over small-scaled oscillations of the phase variance. Thus, the process under consideration can be related to a Wiener process with diffusion coefficient $B_{\rm eff}$. The linear dependence defining the effective diffusion coefficient is found by the least-square method. The next step in our experiment is to measure the ACF of chaotic oscillations of the GIN. Several tens of the

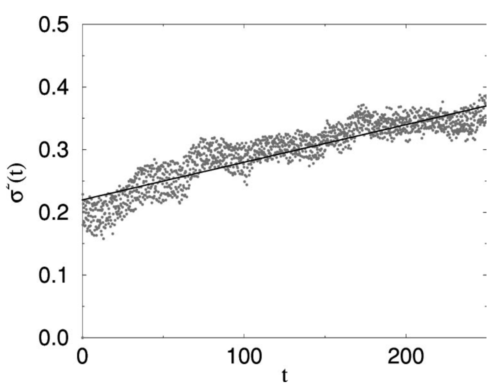

FIG. 2. Temporal dependence of the phase variance in the presence of noise with a root-mean-square value of the noise intensity $D=0.001$ mV and its linear approximation by the least-square method (time t is a dimensionless variable and equals the number of periods of oscillations).

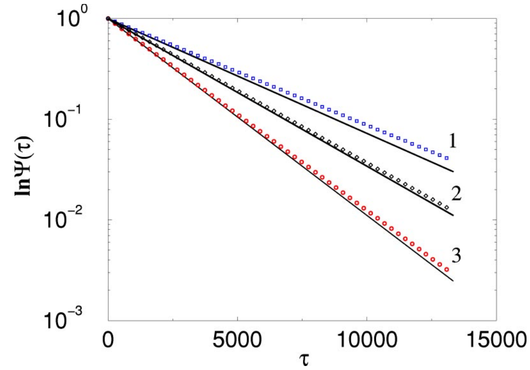

FIG. 3. ACF envelopes (*solid lines*) obtained experimentally for different rms values of the external noise intensity: 1—$D=0$, 2—$D=0.0005$ mV, and 3—$D=0.001$ mV, and their exponential approximations (*dashed lines*) with the decrement of decay $B_{\rm eff}=0.000\,24$, $B_{\rm eff}=0.000\,33$, and $B_{\rm eff}=0.000\,439$, respectively. The other parameters for numeric calculations are $N=100$, $n=262\,144$, and $p=1/2n$.

signal $x(t)$ realizations, each of 10 sec duration, were registered by the fast ADC. The total length of realization is $(3-5)\times 10^5$ oscillation periods with the discretization step Δt corresponding to 37 points per period. The ACF is calculated as follows. First, we compute the time-average value of the x variable for each of the N realizations of the $x(t)$ process:

$$\overline{x}=\frac{1}{n}\sum_{i=1}^{n} x(t_i). \tag{2}$$

Then, we find the mean product $\langle (x(t)x(t+\tau))\rangle$ by averaging over time:

$$K_l(\tau)=\frac{1}{p}\sum_{i=1}^{p} x(t_i)x(t_i+k\Delta t), \quad \tau=k\Delta t_i,$$

$$k=0,1,\ldots,n-p, \tag{3}$$

where $l=1,\ldots,N$ is the number of realization. When calculating one may encounter a problem which is connected with the limitation of a number of $x(t_i)$ values, $i=1,2,\ldots,n$, that are stored in the ADC buffer. The time-averaging result is converged if the number of averagings p is sufficiently large. On the other hand, the greater is the chosen p, the less is the time $\tau_{\rm max}=(n-p)\Delta t$ for the ACF estimation. As the rate of correlation splitting is not high in the regime being considered, the ACF must be computed on a very large time interval. For this reason, the value of p was chosen to be not too large. To attain a high precision of the ACF calculation the obtained data were further averaged over N realizations:

$$\psi(\tau)=\frac{1}{N}\sum_{l=1}^{N} K_l(\tau)-\overline{x^2}. \tag{4}$$

The ACF was normalized on its maximal value at $\tau=0$, i.e., $\Psi(\tau)=\psi(\tau)/\psi(0)$. Figure 3 illustrates logarithmic plots of normalized ACF envelopes that were found experimentally

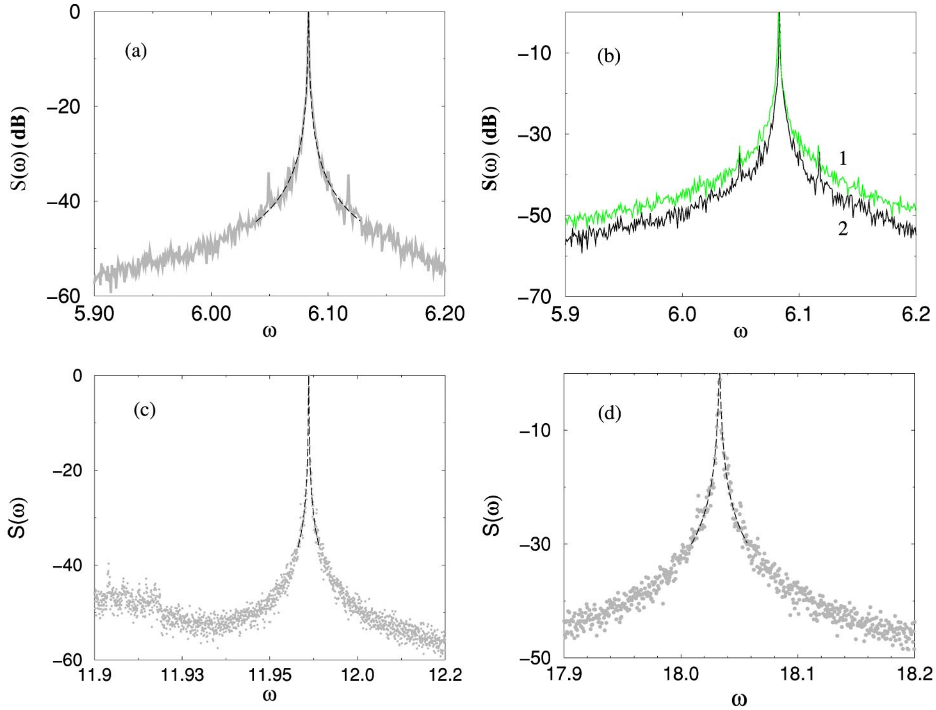

FIG. 4. (a) Experimental power spectrum of the $x(t)$ oscillations in the GIN and its theoretical approximation by Eq. (5) with $B_{\mathrm{eff}}=0.000\,33$ in the presence of noise with $D=0.0005$; (b) power spectra for $D=0.001$ (curve 1) and $D=0$ (curve 2); power spectra at the harmonics $2\omega_0$ with $2.4B_{\mathrm{eff}}$ (c) and $3\omega_0$ with $4B_{\mathrm{eff}}$ (d), where B_{eff} is the experimentally found phase diffusion coefficient (see Fig. 2).

for different values of the external noise intensity. The obtained dependences were approximated according to the exponential law $\Psi_{\mathrm{app}}(\tau)=\exp(-B_{\mathrm{eff}}\tau)$, where B_{eff} is the experimentally found effective diffusion coefficient of the instantaneous phase. The approximation plots are shown in Fig. 3 by symbols. Now let us analyze the results of the power spectrum measurements. The power spectrum of a diffusive process looks like a Lorenzian having the width that is defined by the effective phase diffusion coefficient. For the normalized spectrum the Lorenzian is given by the following expression:

$$S(\omega)=\frac{B_{\mathrm{eff}}}{B_{\mathrm{eff}}^2+(\omega-\omega_0)^2}. \tag{5}$$

In experiment, the effective diffusion coefficient can be independently defined by measuring the spectral peak width. In order to obtain a more precise value of the diffusion coefficient, we approximate the spectral peak with the formula (5) by varying B_{eff}. The resulting value of the coefficient will be the one at which the approximation error is minimal [see Fig. 4(a)].

Figures 4(a) and 4(b) illustrate parts of the experimental power spectra of the GIN both without and in the presence of external noise sources. The spectrum was calculated by means of a standard fast Fourier transform method with averaging. The window length was about 2^{18} points, and the total number of windows was of order 50. The main result is that the effective phase diffusion coefficient values estimated from the spectra are in a good agreement with the values obtained from the linear approximation of the growth of the instantaneous phase variance. The corresponding phase diffusion coefficient values are given in Table I for three different levels of the external noise. Figures 4(c) and 4(d) show spectral maxima at the second and third harmonics of the basic frequency for $D=0$ and their corresponding approximations. The spectral linewidths at the harmonics appear to be significantly larger than the basic linewidth. We have also measured spectral linewidths at subharmonics $n\omega_0/2$. The diffusion coefficient B_{eff} (and the spectral linewidth, respectively) for the subharmonic $n=1$ seems to be less than the corresponding values for the basic frequency ω_0. Our additional numerical calculations with the Rössler system have shown that changes in the B_{eff} values for harmonics and subharmonics are not universal and depend on nonlinear properties of the system. With this, one can only claim that if n increases, the effective diffusion coefficient grows both for harmonics $n\omega_0$ and for subharmonics $n\omega_0/2$ when compared with the value of B_{eff} at the basic frequency. In conclusion, it has been experimentally established that in the regime of spiral chaos the instantaneous phase variance of chaotic oscillations grows, on an average, linearly with the diffusion coefficient B_{eff}. Without noise this coefficient is defined by the chaotic dynamics of the system. In the presence of noise the growth of the phase variance is also linear but the B_{eff} value increases. The ACF of the spiral chaos decays in time according to the exponential law $\exp(-B_{\mathrm{eff}}\tau)$. The spectral linewidth of oscillations at the basic frequency ω_0 is defined

TABLE I. Comparison of phase diffusion coefficient values obtained by different methods without and in the presence of noise with different intensities.

D (mV)	B_{eff} (Hilbert)	B_{eff} (spectrum)
0	0.000244	0.000266
0.0005	0.00033	0.000342
0.001	0.000439	0.000443

by the effective phase diffusion coefficient from the expression (5). This formula can be also applied to measure spectral linewidths at harmonics $n\omega_0$ and subharmonics $n\omega_0/2$. However, in both cases the phase diffusion coefficient values increase with n when compared with the experimental value of B_{eff}.

Therefore, it has been convincingly shown numerically and experimentally that spectral and correlation properties of a wide class of chaotic systems with attractors of the spiral type can be adequately described by the model of a random process of the harmonic noise type.

The work was partly supported by Grant No. SR-006-X1 of the CRDF and the Russian Ministry of Higher Education (Grant No. E02-3.2-345). G.I.S. acknowledges support from INTAS (Grant No. YSF 2002-3/F1a Renewal). V.S.A. and T.E.V. acknowledge support from INTAS Grant No. 01-2061. G.A.O. acknowledges support from INTAS (Grant No. 02-867).

[1] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes in Mathematics, Vol. 470 (Springer, Berlin, 1975).
[2] D. Ruelle, Am. J. Math. **98**, 619 (1976).
[3] Ya. Sinai, in *Nonlinear Waves*, edited by A.V. Gaponov-Grekhov (Nauka, Moscow, 1979), p. 192 (in Russian).
[4] G.M. Zaslavsky, *Chaos in Dynamical Systems* (Harwood Academic, New York, 1985).
[5] A. Arneodo, P. Collet, and C. Tresser, Commun. Math. Phys. **79**, 573 (1981).
[6] V.S. Anishchenko, *Dynamical Chaos—Models and Experiments* (World Scientific, Singapore, 1995).
[7] O.E. Rössler, Phys. Lett. A **57**, 397 (1976).
[8] *Chua's Circuit: A Paradigm of Chaos*, edited by R.N. Madan (World Scientific, Singapore, 1993).
[9] I.D. Farmer, Phys. Rev. Lett. **47**, 179 (1981).
[10] A.S. Pikovsky, Sov. J. Commun. Technol. Electron. **30**, 85 (1985).
[11] M.G. Rosenblum, A.S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).
[12] V.S. Anishchenko, T.E. Vadivasova, A.S. Kopeikin, J. Kurths, and G.I. Strelkova, Phys. Rev. Lett. **87**, 054101 (2001).
[13] V.S. Anishchenko, T.E. Vadivasova, A.S. Kopeikin, J. Kurths, and G.I. Strelkova, Phys. Rev. E **65**, 036206 (2002).
[14] V.S. Anishchenko, T.E. Vadivasova, A.S. Kopeikin, J. Kurths, and G.I. Strelkova, Fluct. Noise Lett. **3**, L213 (2003).
[15] V.S. Anishchenko, T.E. Vadivasova, G.A. Okrokvertskhov, and G.I. Strelkova, Physica A **325**, 199 (2003).
[16] V.S. Anishchenko, T.E. Vadivasova, G.A. Okrokvertskhov, and G.I. Strelkova, Sov. J. Commun. Technol. Electron. **48**, 750 (2003).