Peculiarities of the relaxation to an invariant probability measure of nonhyperbolic chaotic attractors in the presence of noise

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(Received 31 August 2001; published 11 February 2002)

We study the relaxation to an invariant probability measure on quasihyperbolic and nonhyperbolic chaotic attractors in the presence of noise. We also compare different characteristics of the rate of mixing and show numerically that the rate of mixing for nonhyperbolic chaotic attractors can significantly change under the influence of noise. A mechanism of the noise influence on mixing is presented, which is associated with the dynamics of the instantaneous phase of chaotic trajectories. We also analyze how the synchronization effect can influence the rate of mixing in a system of two coupled chaotic oscillators.

DOI: 10.1103/PhysRevE.65.036206 PACS number(s): 05.45.Gg, 05.40.−a

I. INTRODUCTION

It is known that nonlinear dynamical systems can demonstrate both simple oscillations, such as periodic and quasiperiodic, and chaotic oscillations. From a viewpoint of the rigorous theory, hyperbolic chaos is often called ''true'' chaos and is characterized by a homogeneous and topologically stable structure [1–5]. However, strange chaotic attractors of dissipative systems are not, as a rule, robust hyperbolic sets. They are rather referred to as nearly hyperbolic attractors, e.g., the Lorenz attractor. Nearly hyperbolic (quasihyperbolic) attractors include some nonrobust orbits, e.g., separatrix loops, but their appearances and disappearances often do not affect the observed characteristics of chaos, such as a phase portrait, the power spectrum, Lyapunov exponents, and others. Dynamical systems in a chaotic regime may give rise to an invariant measure that does not depend on an initial distribution and fully reflects the statistical properties of the attractor. The existence of an invariant measure has been theoretically proven for hyperbolic and nearly hyperbolic systems [6–11]. Moreover, it has been also established that the white noise of small intensity causes small changes of the structure of stationary distribution. Statistical characteristics of hyperbolic and quasihyperbolic systems are robust to small perturbations [12–15].

However, most chaotic attractors that we deal with in numeric and full-scale experiments are nonhyperbolic [16–18]. The problem of the existence of an invariant measure on a nonhyperbolic chaotic attractor involves serious difficulties because it is generally impossible to obtain a stationary probability distribution being independent of an initial distribution. A nonhyperbolic attractor is a maximal attractor of the dynamical system and encloses a countable set of both regular and chaotic attracting subsets [16,17]. When δ-correlated Gaussian noise is added to the system, an invariant measure on such attractors exists too [19]. But there may be serious problems if the external noise has a finite correlation time and is not Gaussian. The statistical properties of such typical sources of noise present in nonhyperbolic systems will define a number of coexisting attractors, their stationary measure as well as the relaxation time to this measure. The number of attractors being observed and their properties will depend on both the noise statistics and noise intensity. Threshold effects may emerge, which characterize noise-induced transitions [18,20–22]. In the nonhyperbolic case the behavior of phase trajectories is significantly affected by the noise while it changes only slightly in systems with hyperbolic and nearly hyperbolic chaos. Consequently, there is a principal difference to the shadowing problem [13,15,23,24].

A statistical description of noisy nonhyperbolic chaotic attractors is an important and still unsolved problem of the dynamical systems theory. One of the topical problems in this direction is to study the relaxation to stationary distributions in time. There are a number of fundamental questions that have as yet unclear answers. What is a real relaxation time of the system to a stationary distribution? Which factors define this time? Which characteristics can quantify the relaxation time to the stationary measure? What is the role of the noise statistics and the noise intensity in regularities of the relaxation to the stationary distribution? Is there any connection between the relaxation process and the system dynamics?

The relaxation to a stationary distribution, if the latter exists, is described by the evolutionary equations. If the noise source is normal and uncorrelated, this process is determined by the Fokker-Planck equation (FPE) or the Frobenius-Perron equation. However, if the dynamical system is high dimensional ($N \geqslant 3$), the nonstationary solution of the FPE is difficult enough to find even numerically. Thus, in our studies we use the method of stochastic differential equations [22]. The relaxation to the invariant probability measure is related to the mixing. In chaotic systems that have an invariant probability measure [6,7,3,10,8,9] the properties of mixing can be characterized by the Kolmogorov entropy H_{K} [25,26] that determines the characteristic time of mixing, $\tau_{\mathrm{mix}} = H_{\mathrm{K}}^{-1}$. For invertible maps that satisfy the Smale axiom, τ_{mix} defines the rate of exponential decay of the autocorrelation function. It has been proven that [3]

$$\Psi(\tau) = \exp(-H_{\mathrm{K}}\tau). \tag{1}$$

This means that the correlation time τ_{cor} coincides with τ_{mix}.

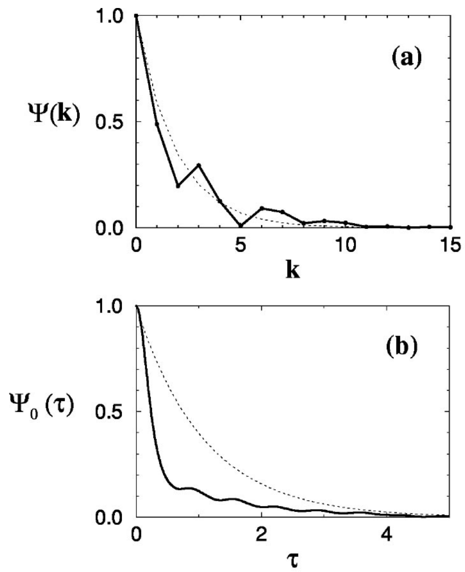

FIG. 1. Autocorrelation function (solid line) and its theoretical approximation (1) (dotted line) for (a) the Lozi map $x_{n+1}=1-a|x_n|+y_n$; $y_{n+1}=bx_n$ at $a=1.75$, $b=0.3$ and (b) the Lorenz attractor for $r=28$, $\sigma=10$, and $b=8/3$.

At the same time, when the conditions of strong hyperbolicity are fulfilled, the Kolmogorov entropy is defined by the equality

$$H_{\mathrm{K}}=\sum_{j=1}^{k}\lambda_j^+, \tag{2}$$

where λ_j^+ are the positive Lyapunov exponents. In more general cases, the upper bound is valid, i.e., $H_{\mathrm{K}}\leqslant\Sigma\lambda_j^+$ [26,27,11,9].

As has been noted above, a stationary distribution and consequently, a probability measure on a nonhyperbolic chaotic attractor without the noise are not defined. The equality (2) is not valid for nonhyperbolic attractors. Thus, the problem of the interconnection among the time of mixing, the correlation time, and the positive Lyapunov exponent in differential systems and maps still remains unresolved. Figure 1 shows theoretical (1) and numerical results of the autocorrelation function for the Lozi map [28] and for the Lorenz attractor. We observe that the experimental and theoretical estimates for the Lozi map [Fig. 1(a)] are in good agreement with each other and thus verify the conclusions of Pesin's theorem [27]. However, in the case of the Lorenz attractor [Fig. 1(b)] the experimental dependence cannot be approximated by formula (1) and is characterized by a smaller correlation time. This result testifies that exponential estimates for the rate of correlations decreasing as well as for the mixing time are not always valid for flow systems [29,30], even in the nearly hyperbolic case.

The mixing on a chaotic attractor of a flow system in $\mathbb{R}^3$ can be represented as a superposition of two processes if we introduce an appropriately chosen Poincaré section. The first component is the mixing produced by a map on the two-dimensional Poincaré section, whereas the second one depends on the motion of the phase point along the trajectory outside of the Poincaré section. The mixing in the map must rather satisfy Eq. (1) and be defined by the positive Lyapunov exponent [for example, see Fig. 1(a)]. The second component is the mixing process along the flow of the phase trajectories and depends on the properties of the phase trajectories motion between the consecutive returns to the Poincaré section.

In this paper we study numerically the relaxation to a stationary measure on nonhyperbolic attractors of the different structure in the presence of noise. We attempt to answer at least a part of the questions stated above. In particular, we pay our attention to the following problems.

(1) The inter-relation between the relaxation rate of a chaotic system to a stationary probability distribution, the autocorrelation function and the Kolmogorov entropy in the case of nearly hyperbolic and nonhyperbolic attractors.

(2) The influence of noise on the rate of relaxation and on the characteristics of mixing for different types of chaos.

This paper develops and supplements the main results recently reported in Ref. [31]. The organization of the paper is as follows. In Sec. II we introduce the models that we are dealing with and describe the numeric methods. Section III is devoted to the study of the relaxation to a stationary distribution on both a quasihyperbolic and a nonhyperbolic attractor in the Lorenz system. In both cases we investigate the role of external noise. In Sec. IV we consider the relaxation process for two different nonhyperbolic chaotic attractors in the Rössler system. We compare the results obtained in Secs. III and IV and present a mechanism of the noise influence on the rate of relaxation to a stationary distribution. In Sec. V we examine the relaxation process in a system of two interacting Rössler oscillators. And finally, we summarize the obtained numeric results and formulate conclusions in Sec. VI.

II. MODELS AND NUMERIC METHODS

We investigate chaotic attractors in paradigmatic continuous-time systems both in the absence and in the presence of noise. In particular, we consider the Lorenz system [32]

$$\begin{aligned}\dot{x}&=-\sigma(x-y)+\sqrt{2D}\xi(t),\\ \dot{y}&=rx-y-xz,\\ \dot{z}&=-\beta z+xy,\end{aligned} \tag{3}$$

and the Rössler oscillator [33]

$$\begin{aligned}\dot{x}&=-y-z+\sqrt{2D}\xi(t),\\ \dot{y}&=x+ay,\\ \dot{z}&=b-mz+xz.\end{aligned} \tag{4}$$

In both models $\xi(t)$ is a normal white noise source[1] with the mean value $\langle \xi(t) \rangle \equiv 0$ and correlation $\langle \xi(t)\xi(t+\tau) \rangle \equiv \delta(\tau)$, where $\delta()$ is Dirac's function. The parameter D denotes the noise intensity. In the Lorenz system we choose two different regimes, namely, a quasihyperbolic attractor ($\sigma=10$, $\beta=8/3$, and $r=28$) and a nonhyperbolic attractor ($\sigma=10$, $\beta=8/3$, and $r=210$). For the Rössler system we fix $a=0.2$ and $b=0.2$ and vary the control parameter m in the interval [4.25,13]. We integrate Eqs. (3) and (4) using a fourth-order Runge-Kutta routine with fluctuations taken into account. Chaotic attractors of systems (3) and (4) have been studied in detail and are typical examples of quasihyperbolic and nonhyperbolic chaos. Thus, results obtained for Eqs. (3) and (4) can be generalized to a wide class of dynamical systems [34,35].

To examine the relaxation to a stationary distribution in these systems, we analyze how points situated at an initial time in a cube of small size δ around an arbitrary point of the trajectory belonging to an attractor of the system evolve with time. We take $\delta=0.09$ for the size of this cube and fill it uniformly with $n=9000$ points. As time goes on, these points in the phase space are distributed throughout the whole attractor. To characterize the convergence to the stationary distribution we follow the temporal evolution of this set of points and calculate the ensemble average

$$\overline{x(t)}=\int_W p(x,t)x\,dx=\frac{1}{n}\sum_{i=1}^{n} x_i(t). \tag{5}$$

Here, x is one of the system dynamical variables, and $p(x,t)$ is the probability density of the variable x at the time t, which corresponds to the chosen initial distribution. It is known that the phase trajectory of system (3) visits neighborhoods of two saddle foci. In this case, when calculating $\overline{x(t)}$ one may first sum separately over points having fallen in the neighborhood of each saddle focus, and then combine the obtained results. However, the mean value appears to approach zero in a short time interval and its further evolution is badly detected. To follow the relaxation in Eq. (3) we compute the mean value when points in the neighborhood of only one saddle focus are taken into account. In this case the relaxation to this quantity goes more slowly in time. Then we calculate the function $\gamma(t_k)$,

$$\gamma(t_k)=|\overline{x}_m(t_{k+1})-\overline{x}_m(t_k)|, \tag{6}$$

where $\overline{x}_m(t_k)$ and $\overline{x}_m(t_{k+1})$ are successive extrema of $\overline{x}(t)$. Thus, $\gamma(t_k)$ characterizes the amplitude of the mean value oscillations (see Fig. 2). In Eq. (6) t_k and t_{k+1} are successive time moments corresponding to the extrema of $\overline{x}$. The temporal behavior of $\gamma(t_k)$ allows to judge the character and the rate of relaxation to the probability measure on the attractor.

[1]The noise source is added to only one equation of the dynamical systems (3) and (4) as the inclusion of noise in all three equations does not change qualitatively the results but significantly retards integration.

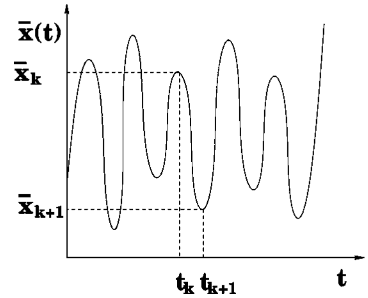

FIG. 2. Schematic illustration of the calculation of $\gamma(t_k)$ Eq. (6).

We can estimate the time T_ε in the course of which the stationary probability density can be obtained with a given accuracy ε, i.e., when the inequality $\gamma(t_k)<\varepsilon$ is fulfilled for any $t>T_\varepsilon$, and ε is small. However, this characteristic has some disadvantages, one of which is that T_ε depends on the initial magnitude of the phase space element that we follow. Our numeric calculations have shown that for the attractors under study either $\gamma(t_k)$ or its envelope can be approximated by the exponential law $\gamma(t_k)\approx\exp(-t/\alpha)$. The exponent α denotes the time, in which the function graph decreases in e times, and characterizes the rate of mixing. The value of α does not depend on the size of the initial phase space element but significantly increases with the increasing noise intensity D. This quantity appears to be more suitable for estimating the rate of mixing.

We also calculate the maximal Lyapunov exponent (LE) λ_1 of a chaotic trajectory on an attractor. For hyperbolic attractors in $\mathbb{R}^3$ λ_1 is positive, has the same value for all typical phase trajectories and represents an averaged characteristic of the rate of mixing. In the nonhyperbolic case, the maximal LE, as well as other characteristics computed along a single trajectory, may depend on the choice of this trajectory. However, since a "computer noise" is inevitably present in numeric experiments, such a dependence cannot be always detected. Besides, we also compute the normalized autocorrelation function of steady-state oscillations $x(t)$,

$$\Psi(\tau)=\left|\frac{\langle x(t)x(t+\tau)\rangle-\overline{x}(t)\overline{x}(t+\tau)}{\sqrt{[\langle x^2(t)\rangle-\overline{x}^2(t)][\langle x^2(t+\tau)\rangle-\overline{x}^2(t+\tau)]}}\right|. \tag{7}$$

To make some figures more informative and compact, instead of $\gamma(t_k)$ and $\Psi(\tau)$ we plot (where it is necessary) their envelopes $\gamma_0(t_k)$ and $\Psi_0(\tau)$, respectively.

III. RELAXATION TO A PROBABILITY MEASURE IN THE LORENZ SYSTEM

We start with considering chaotic attractors in the Lorenz system and are interested in knowing as to how additive noise influences the relaxation to their stationary distributions. Figure 3 shows the behavior of $\gamma_0(t_k)$ for both quasihyperbolic and nonhyperbolic chaotic attractors of Eqs. (3) with and without noise added. We find that for the Lorenz attractor noise does not significantly influence the relaxation

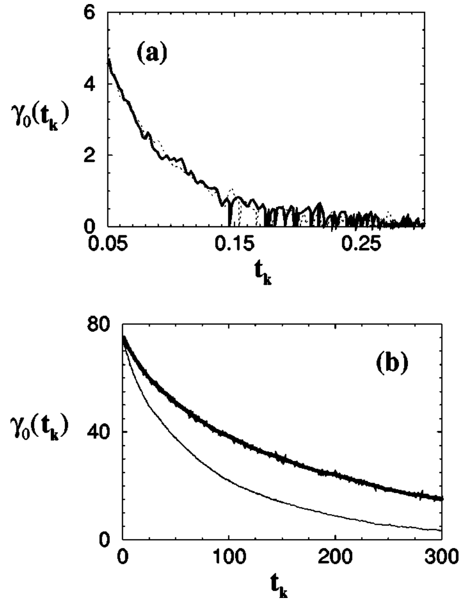

FIG. 3. $\gamma_0(t_k)$ for chaotic attractors in the Lorenz system (3). (a) For $r=28$ and $D=0$ (solid line), and $D=0.01$ (dotted line); (b) for $r=210$ and $D=0$ (thick line), and for $r=210$ and $D=0.01$ (thin line).

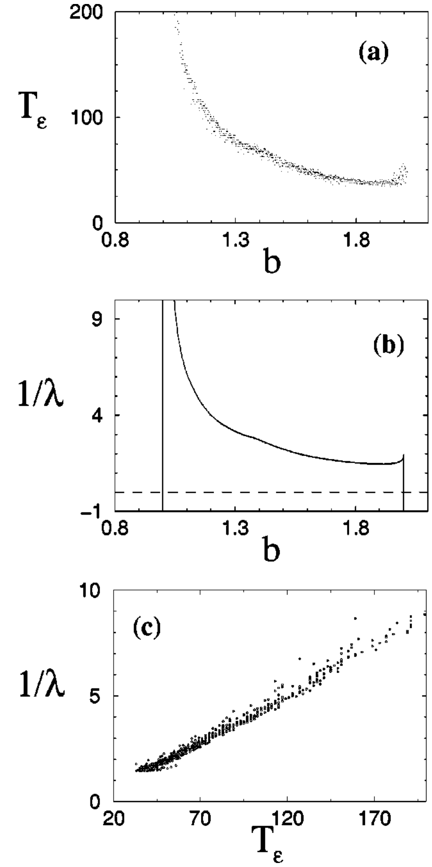

FIG. 4. For the model map (8), (a) T_ε as a function of the control parameter b; (b) $1/\lambda$ versus b, and (c) $1/\lambda$ as a function of T_ε.

rate [Fig. 3(a)]. However, we observe a quite different situation for the nonhyperbolic attractor. There the rate of relaxation is strongly affected by noise [Figs. 3(b) and 3(c)] and the characteristic time α decreases more than twice under the influence of noise (see Table I). Besides, the time of relaxation to the stationary distribution of the nonhyperbolic attractor [Figs. 3(b,c)] is abruptly increased as compared with that one for the quasihyperbolic attractor [Fig. 3(a)].

If the characteristic time of mixing satisfies the relation $\tau_{\text{mix}} \sim 1/\lambda^+$, then we may assume that T_ε, α and $1/\lambda^+$ are directly proportional to each other. To simplify our computations of T_ε and α as functions of the system (3) parameter, we use a one-dimensional map that simulates the return map in a Poincare section of the Lorenz attractor ([36]),

$$x_{n+1}=\begin{cases} 1-b|x_n|^a, & x_n\in[-1,0) \\ 0, & x_n=0 \\ -1+b|x_n|^a, & x_n\in(0,1], \end{cases} \tag{8}$$

where $a=1/b+0.001$. Figures 4(a,b) shows the dependences of T_ε and $1/\lambda$ on the control parameter b in the region of positive λ. It is seen that both functions are quite similar. For illustrative purposes we also plot T_ε versus $1/\lambda$ in Fig. 4(c). There points are arranged along a straight line whose slope defines the coefficient of proportionality K. Thus, the rate of mixing in the model map is unambiguously determined by the λ^+. This result is in complete agreement with theoretically proven statements for hyperbolic one-to-one two-dimensional maps, although map (8) does not generally belong to this class.

As we have shown for the nonhyperbolic attractor in system (3), the rate of mixing predicted by means of α can

TABLE I. The exponent α, the largest LE λ_1 and the correlation time τ_{cor} for attractors in the Lorenz system ($\sigma=10,\beta=8/3$) and the Rössler system ($a=0.2,b=0.2$).

Lorenz system					Rössler system				
r	D	α	λ_1	τ_{cor}	m	D	α	λ_1	τ_{cor}
28	0	0.056	0.92	0.4	6.1	0	470	0.082	9500
28	0.01	0.056	0.92	0.4	6.1	0.001	330	0.081	5500
210	0	165	0.86	25000	6.1	0.1	110	0.081	200
210	0.01	78	0.86	13000	13	0	40	0.11	40
					13	0.01	45	0.11	40

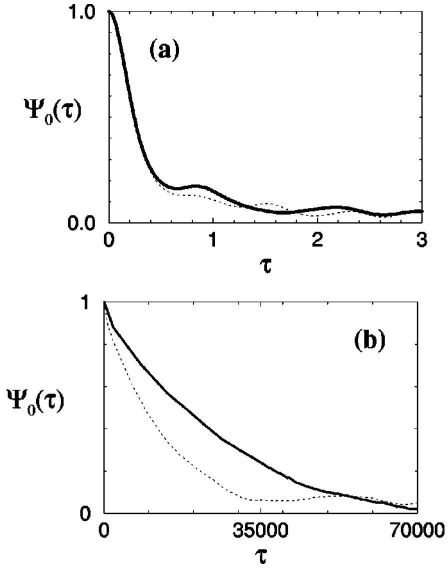

FIG. 5. Envelopes of the normalized autocorrelation function $\Psi_0(\tau)$ for attractors in system (3). (a) $r=28$ and $D=0$ (solid line), and $D=0.01$ (dotted line); (b) $r=210$, $D=0$ (solid line), and $D=0.01$ (dotted line).

considerably change under the influence of noise. Now we are going to check whether the other characteristics of the mixing rate, such as the LE and the correlation time, will also depend on noise perturbations. For the same chaotic attractors in the Lorenz system we compute the largest LE λ_1 and estimate the normalized autocorrelation function $\Psi(\tau)$, $\tau=t_2-t_1$, of the dynamical variable $x(t)$ for different noise intensities D. The values of λ_1 and $\tau_{\rm cor}$ are given in Table I. We find that for both types of chaotic attractors the LE does not depend, within the calculation accuracy, on the noise intensity. The autocorrelation function of the quasihyperbolic attractor is practically not affected by noise [see curves 1 and 2 in Fig. 5(a)]. However, in the regime of a nonhyperbolic attractor it decreases more rapidly in the presence of noise [see curves in Figs. 5(a,b)]. Our calculations show that the correlation time $\tau_{\rm cor}$ for the quasihyperbolic chaos does not depend on the noise level, whereas for the nonhyperbolic attractor it decreases almost twice under the influence of noise (see Table I).

Thus, on the one hand, the maximal LE of both chaotic attractors remains almost unchanged in the presence of noise. On the other hand, while for the quasihyperbolic attractor $\tau_{\rm cor}$ and α are practically insensitive to fluctuations, these characteristics for the nonhyperbolic attractor change considerably under the influence of noise.

IV. RELAXATION TO A STATIONARY DISTRIBUTION IN THE RÖSSLER SYSTEM: MECHANISM OF THE EFFECT OF NOISE ON THE RATE OF MIXING

In the preceding section we have analyzed two different chaotic attractors in the Lorenz system and found a quite different effect of noise on the relaxation process. In this section we try to explain a mechanism of this difference.

Since the statistical characteristics of hyperbolic and quasihyperbolic attractors are known to be stable to noise perturbations [7,13–15,19], we can predict that characteristics of the mixing rate for this type of attractor will be insensitive to the influence of noise. On the contrary, in a regime of nonhyperbolic attractor the noise affects significantly the chaotic system behavior. Thus, in this case noise can change essentially the mixing characteristics. To verify this statement we analyze nonhyperbolic attractors of different types. The nonhyperbolic attractor in the Lorenz system represents a frequently encountered type of nonhyperbolic attractors, called spiral attractor [37,34,38]. It is usually generated through an infinite sequence of period-doubling bifurcations. A chaotic attractor realizing in the Rössler system (4) at fixed $a=b=0.2$ and in the parameter m interval $[4.25,8.5]$ serves as a well-known example of a spiral attractor. The phase trajectory on the spiral attractor rotates with a high regularity around one or several saddle foci. The autocorrelation function is oscillating and the power spectrum exhibits narrow-band peaks corresponding to the mean rotation frequency, its harmonics, and subharmonics. By virtue of these properties spiral chaos is called phase coherent [33,39,40].

The chaotic attractor of Eq. (4) is qualitatively changing as the parameter m increases. In the interval $8.5<m<13$ there occurs a nonhyperbolic attractor of noncoherent type, called funnel attractor [37,34]. Phase trajectories on the funnel attractor make complicated loops around a saddle focus and thus, demonstrate a nonregular rotation behavior. Consequently, the autocorrelation function of the noncoherent chaos decreases much rapidly than that in the coherent case, and the power spectrum does not already contain sharp peaks.

The calculations performed for $m\in[4.25,7.5]$ (spiral chaos) and for $m\in[8.5,13]$ (noncoherent chaos) allow to assume that an invariant probability measure exists for the parameter values considered. All the effects being observed for each type of attractor in Eq. (4) are qualitatively preserved when the parameter m is varied. In our numeric simulation we fix $m=6.1$ for the spiral attractor and $m=13$ for the funnel attractor.

Figure 6 shows the typical behavior of $\gamma_0(t)$ for both the spiral and the funnel attractor of the Rössler system. We find that, as in the case of the spiral attractor in the Lorenz system, the noise significantly influences the rate of mixing in the regime of spiral attractor in the Rössler system. The value of α is strongly decreasing for the increasing noise intensity [see Table I and Fig. 6(a)].

We find a quite different situation for the funnel attractor. Noncoherent chaos is practically insensitive to noise perturbations. Both α and λ_1 do not significantly change when noise is added to Eqs. (4). At the same time, it is well known that noncoherent chaos exhibits a close similarity to random processes. This fact can be verified, e.g., by means of the autocorrelation function $\Psi(\tau)$ for the spiral and the funnel attractors in system (4) (Fig. 7). Our numerical experiments show that the correlation times are essentially different for these two chaotic regimes (see Table I); without noise they differ by two orders. On the one hand, in the case of coherent chaos the correlation time decreases dramatically in the pres-

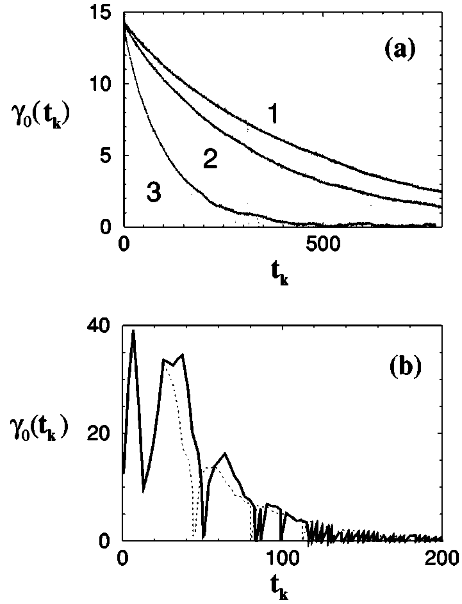

FIG. 6. $\gamma_0(t_k)$ for attractors in the Rössler system (4). (a) For the spiral attractor ($m=6.1$) for $D=0$ (curve 1), $D=0.001$ (curve 2), and $D=0.1$ (curve 3); (b) for the funnel attractor ($m=13$) for $D=0$ (solid line) and for $D=0.01$ (dotted line).

ence of noise [Fig. 7(a) and Table I). On the other hand, the autocorrelation function for the funnel attractor in the deterministic case practically coincides with that in the presence of noise [Fig. 7(b)]. Hence, noncoherent chaos, which is nonhyperbolic, demonstrates some property of hyperbolic chaos, i.e., "dynamical stochasticity" turns out to be much stronger than that imposed from an external (additive) one [7]. This experimental result is interesting and requires a more detailed consideration. It is also worth noting another finding of our simulations. In the regime of spiral chaos the rate of

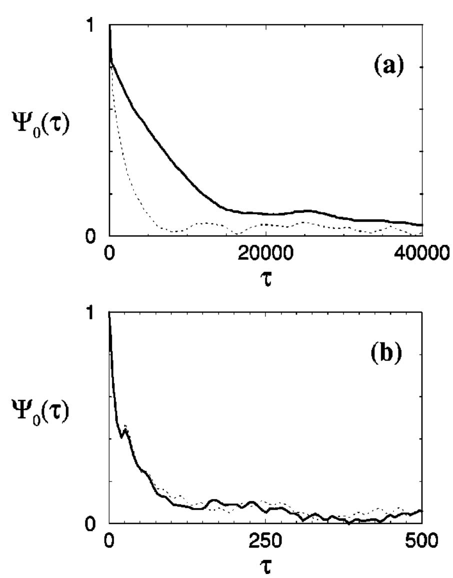

FIG. 7. Envelopes of the normalized autocorrelation function $\Psi_0(\tau)$ for attractors in Eq. (4). (a) At $m=6.1$ and for $D=0$ (solid line) and $D=0.01$ (dotted line); (b) at $m=13$ for $D=0$ (solid line) and $D=0.01$ (dotted line).

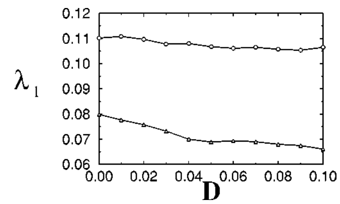

FIG. 8. For the Rössler system, λ_1 on the spiral (*triangles*) and the funnel (*circles*) attractor as functions of the noise intensity *D*.

mixing is not uniquely determined by the largest LE but depends strongly on the noise intensity. This result does not match theoretical conclusions [25,26] obtained for hyperbolic chaos in two-dimensional maps.

We have found that the positive LE is weakly sensitive to fluctuations (see Fig. 8), and rather grows not much with the increasing noise intensity, whereas in certain cases the correlation time changes considerably under the influence of noise. Thus, the positive Lyapunov exponents are not the unique characteristic responsible for the mechanism of mixing on a nonhyperbolic attractor. We suppose that the essential effect of noise on relaxation to the stationary distribution may be associated with peculiarities of the phase trajectory dynamics in the neighborhood of an unstable equilibrium state. Since the trajectory rotates almost regularly on the spiral attractor, the relaxation process appears to be very long. The addition of noise to the system destroys the relative regularity of the trajectory and, consequently, the rate of mixing significantly increases.

It is known that for chaotic oscillations one can introduce the notion of instantaneous amplitude and phase [41]. The instantaneous phase characterizes the rotation of a trajectory around a saddle focus. System (4) is of such type because the trajectory in the (x-y) projection rotates around the unique saddle focus located very near to the origin. To quantify the trajectory dynamics in this case, the instantaneous phase can be introduced as follows:

$$\Phi(t)=\arctan\frac{y(t)}{x(t)}+\pi n(t), \tag{9}$$

where $n(t)=0,1,2,\ldots$ is the number of intersections of the phase trajectory with the plane $x=0$.

We consider the instantaneous phase difference $\Delta_n=\Phi_2(t_n)-\Phi_1(t_n)$ of two initially close trajectories of system (4) as a function of the time. We again find that in the regime of spiral chaos [Fig. 9(a)], noise drastically changes the temporal behavior of the phase differences $\Delta\Phi$. When $D=0$, the phase difference varies slowly, with the exception of fine-scaled changes within $\pm\pi$. However, the addition of noise leads to changes of $\Delta\Phi$ much larger than 2π. Thus, mixing is strongly enhanced under the influence of noise. It is important to emphasize that phase changes are very typical for the non-coherent attractor already in a purely deterministic case. Therefore, the variations of Δ_n are qualitatively the same with and without the presence of noise [see Fig. 9(b)]. The component of mixing along the flow of trajectories is related to the divergence of the instantaneous phase values and thus, is determined by the temporal behavior of

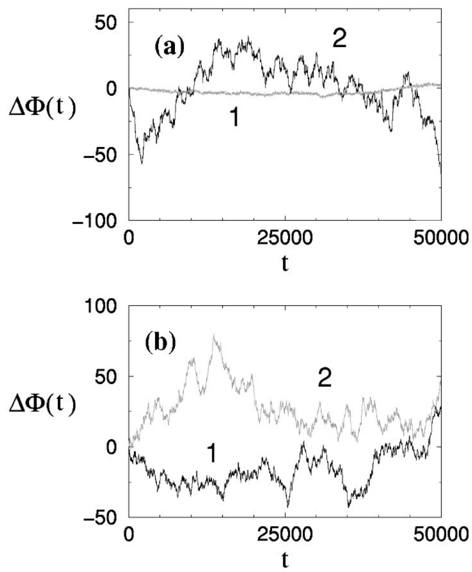

FIG. 9. Instantaneous phase difference $\Delta\Phi$ on t for $m=6.1$ (a) and $m=13$ (b) in the noise-free case (curves 1) and in the presence of noise with intensity $D=0.1$ (curves 2).

the phases. The instantaneous phase of an ensemble of initially close trajectories on the spiral attractors remain very close to each other over a long period of time, although the points in the secant plane are spread over the whole attractor section. In this case the relaxation to a stationary probability distribution on the whole attractor of a flow system will be much longer than that in the Poincaré map. The violation of regular rotation of trajectories is characteristic for the funnel attractor and leads to a nonmonotonic dependence of the instantaneous phase on the time. The phase trajectory creates complicated loops at nonequal time intervals that causes the value of the current phase to decrease slightly. This results in a rapid divergence of the phase values of neighboring trajectories. The influence of noise on spiral chaos leads to similar effects. Figure 10(a) shows the temporal dependences of the variation of the instantaneous phase σ_Φ^2 on an ensemble of initially close trajectories for both the spiral and the funnel attractor of system (4). We observe that in both the noisy and the noise-free case the variation grows almost linearly on the time intervals being considered. However, in the case of spiral chaos without noise (curve 1), the value of σ_Φ^2 is small (on the given time interval it does not exceed the variation of the uniform phase distribution on the interval $[-\pi;\pi]$) and increases much slower than in the other cases considered. The linear growth of the variation allows to estimate the divergence of the instantaneous phases by using the effective diffusion coefficient

$$D_{\mathrm{eff}}=\frac{1}{2}\,\frac{d\sigma_\Phi^2(t)}{dt}. \tag{10}$$

Figure 10(b) illustrates the dependences of D_{eff} of the instantaneous phase of chaotic oscillations on the noise intensity for both the spiral and the funnel attractor in the Rössler system (4). It is seen that in both cases D_{eff} grows with increasing D but for spiral chaos this growth is more significant. This result strongly testifies that D_{eff} is a very effective characteristic for diagnosing the statistical properties of a chaotic attractor in the presence of fluctuations.

Well-known quasihyperbolic attractors in three-dimensional continuous-time systems, such as the Lorenz attractor, the Morioka-Shimizu attractor [42], are attractors of the switching type. The phase trajectory switches chaotically from the neighborhood of one saddle equilibrium state to the neighborhood of another one. Such switchings are accompanied by chaotic phase changes even without noise. In this case the addition of noise does not change considerably the phase dynamics and, consequently, does not influence the rate of relaxation to the stationary distribution.

V. MIXING IN COUPLED RÖSSLER SYSTEMS

When two identical chaotic oscillators interact, there occurs the phenomenon of complete chaotic synchronization [43–46]. It manifests itself in a complete coincidence of chaotic oscillations of the partial systems. In the regime of complete synchronization a chaotic attractor belongs to the invariant manifold U which is defined by the symmetry relations $\mathbf{X}_1=\mathbf{X}_2$, where $\mathbf{X}_1$ and $\mathbf{X}_2$ are vectors of the dynamical variables of the first and the second system, respectively. For attractors in U, the instantaneous phase difference of oscillations, introduced in [41], is constant and identically equal to zero. Such attractors will be called in phase and

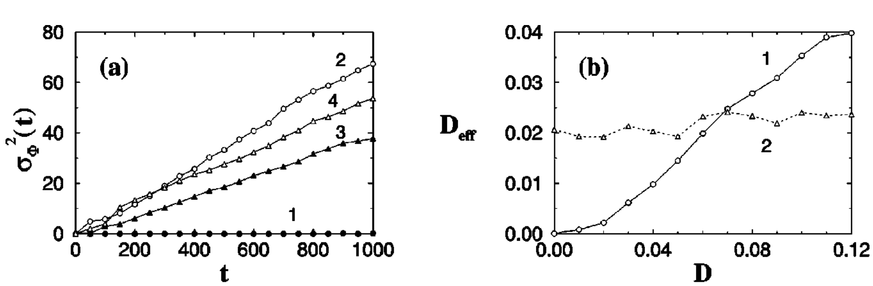

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FIG. 10. Characteristics of the instantaneous phase divergence of neighboring trajectories for spiral chaos ($m=6.1$) and funnel chaos ($m=13$) in Eqs. (4). (a) Temporal dependences of the variation of the instantaneous phase σ_Φ^2 for spiral chaos at $D=0$ (curves 1), $D=0.1$ (curves 2), and for noncoherent chaos at $D=0$ (curves 3), $D=0.1$ (curves 4); (b) The effective diffusion coefficient D_{eff} as a function of the noise intensity D for spiral (curves 1) and noncoherent (curves 2) chaos.

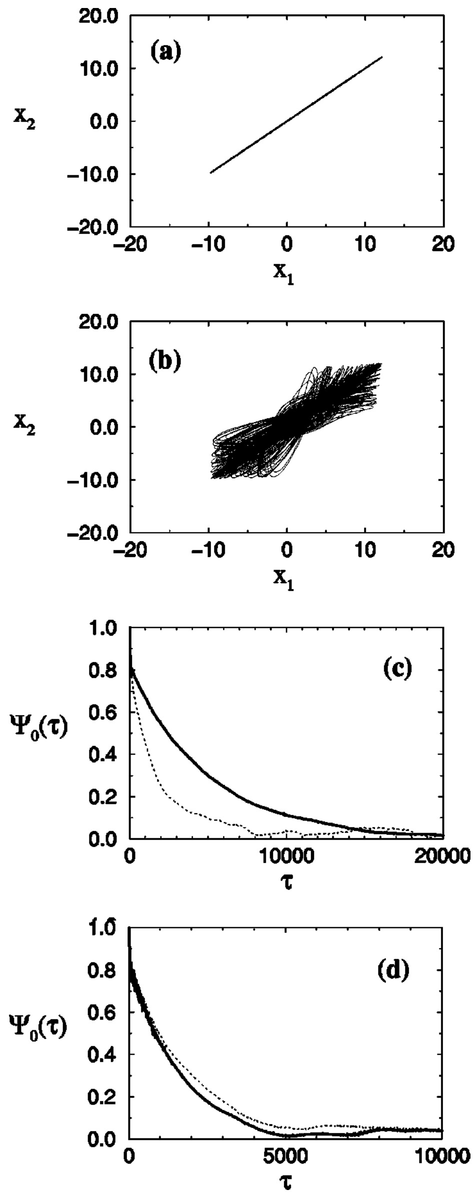

FIG. 11. For system (11) at $m=6.1$, projections of the in-phase ($\alpha=0.5$) (a) and the out-of-phase ($\alpha=0.05$) (b) attractor and envelopes of their autocorrelation functions (c) and (d) for $D=0$ (solid line) and $D=0.01$ (dotted line). The other parameters are $a=b=0.2$ and $\omega=0.97$.

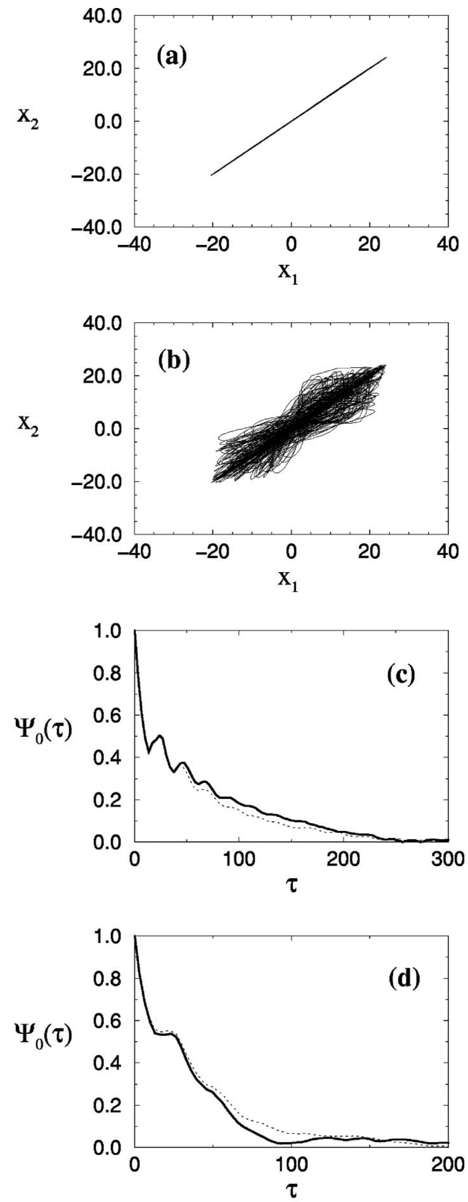

FIG. 12. For system (11) at $m=13$, projections of the in-phase ($\alpha=0.5$) (a) and the out-of-phase ($\alpha=0.1$) (b) attractor and envelopes of their autocorrelation functions (c) and (d) for $D=0$ (solid line) and $D=0.01$ (dotted line). The other parameters are $a=b=0.2$ and $\omega=0.97$.

attractors which do not lie in U will be called out of phase. An in-phase chaotic attractor is topologically similar to an attractor of the partial system and its structure is more simple than that of an out-of-phase attractor. Accordingly, we may assume that the in-phase and the out-of-phase attractors have a different rate of mixing. Moreover, they can be characterized by a different influence of noise on the relaxation to an invariant measure.

To find how the synchronization effect can influence the rate of mixing, we investigate a system of two mutually coupled identical Rössler oscillators that behave chaotically. The parameters of both oscillators were chosen the same. The system equations read

$$
\begin{aligned}
\dot{x}_1&=-\omega y_1-z_1+\alpha(x_2-x_1)+\sqrt{2D}\xi_1(t),\\
\dot{y}_1&=\omega x_1+a y_1,\\
\dot{z}_1&=b-m z_1+x_1 z_1,\\
\dot{x}_2&=-\omega y_2-z_2+\alpha(x_1-x_2)+\sqrt{2D}\xi_2(t),\\
\dot{y}_2&=\omega x_2+a y_2,\\
\dot{z}_2&=b-m z_2+x_2 z_2,
\end{aligned}
\tag{11}
$$

where α is the coupling parameter and $\xi_1(t)$ and $\xi_2(t)$ are independent white noise sources. We compute the normalized autocorrelation function for the dynamical variable $x_1(t)$.

The (x_1-x_2) projections of in-phase and out-of-phase chaotic attractors and the envelopes of their autocorrelation functions both with and without noise added are shown in Figs. 11 and 12 for spiral and noncoherent chaos in the partial oscillator of system (11), respectively.

We find that in the regime of complete in-phase synchronization the correlation time is significantly larger than that in the out-of-phase regime at the same value. Additional phase changes can occur when in-phase synchronization is destroyed and the structure of the chaotic attractor gets more complicated [47]. In this case the correlation time decreases and, consequently, the rate of mixing increases. Such behavior of the correlation time in the synchronization regime is observed both for the spiral and the funnel attractor in the partial system. It is worth noting that, as in the previous cases, noise influences the correlation time only for systems with phase coherent dynamics, i.e., for coupled systems with a spiral attractor in the regime of complete synchronization. In cases when phase changes occur already in a purely deterministic case, the effect of noise appears to be minor. These results are in complete agreement with those ones previously obtained for a single oscillator.

VI. CONCLUSIONS

In this paper we have shown that for nearly hyperbolic attractors the characteristics of the mixing rate are in good correspondence with each other and practically do not depend on the noise intensity.

However, there is a group of nonhyperbolic attractors of spiral type for which noise strongly influences the characteristics of the relaxation to a stationary distribution as well as the correlation time and practically does not change the positive Lyapunov exponent.

The rate of mixing on nonhyperbolic attractors in $\mathbb{R}^3$ is determined not only by the positive Lyapunov exponent but also depends on the instantaneous phase dynamics of chaotic oscillations. In the regime of spiral chaos, noise causing phase changes can essentially accelerate the relaxation to a stationary distribution. However, for chaotic attractors with a nonregular behavior of the instantaneous phase the rate of mixing cannot be considerably affected by noise. This statement has been checked and verified for different types of chaotic attractors with nonregular behavior of the instantaneous phase, namely, for quasihyperbolic attractors, nonhyperbolic attractors of noncoherent type and out-of-phase chaotic attractors in interacting systems.

ACKNOWLEDGMENTS

We are grateful to Dr. D. Turaev for valuable discussions on a number of mathematical problems. This work was partly supported by Award No. REC-006 of the CRDF and the RFBR (Grant No. 00-02-17512), SFB 555 (DFG, Germany) and EU RTN 158. T.E.V. acknowledges support from INTAS (Grant No. 01-867). V.S.A. acknowledges support from the Humboldt Foundation.

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