

INSTANTANEOUS PHASE METHOD IN STUDYING CHAOTIC AND STOCHASTIC OSCILLATIONS AND ITS LIMITATIONS

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We study the behavior of an instantaneous phase and mean frequency of chaotic self-sustained oscillations and noise-induced stochastic oscillations. The results obtained by using various methods of the phase definition are compared to each other. We also compare two methods for describing synchronization of chaotic self-sustained oscillations, namely, instantaneous phase locking and locking of characteristic frequencies in power spectra. It is shown that the technique for diagnostics of the chaos synchronization based on the instantaneous phase locking is not universal.

Keywords: Chaotic and stochastic oscillations; spiral and funnel chaos; instantaneous phase; mean frequency; synchronization.

1. Introduction

In recent years, the “phase dynamics” method for studying synchronization effects in chaotic [1–4] and stochastic [3–9] systems have received wide acceptance. In this method, the synchronization phenomenon is treated as an effect of instantaneous phase locking of oscillations introduced into consideration in some way [2–4]. In many cases, the studies of the instantaneous phase dynamics of aperiodic oscillations lead to important and interesting results [1–4, 7, 9]. However, it is necessary to carefully take into account the fact that the definition of the instantaneous phase cannot be introduced uniquely [2–4]. The most general definition of the instantaneous phase of irregular oscillations is the one used in the theory of stochastic processes [1, 2, 10]. For oscillations $x(t)$ (with zero mean $\langle x(t) \rangle \equiv 0$), Hilbert-conjugate process $x_h(t)$ is introduced as

$$x_h(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau} d\tau, \quad (1)$$

where the improper integral is understood in terms of its principal value. If integral (1) exists, we can write

$$x(t) + ix_h(t) = A(t) \exp(i\Phi(t)), \quad (2)$$

where $i = \sqrt{-1}$, $A(t)$ is the instantaneous amplitude (envelope), and $\Phi(t)$ is the instantaneous phase of oscillations $x(t)$. For the instantaneous phase we have

$$\Phi(t) = \operatorname{arctg} \left(\frac{x_h(t)}{x(t)} \right) \pm \pi k, \quad k = 0, 1, 2, \dots \quad (3)$$

The choice of integer k is determined by the continuity condition for function $\Phi(t)$. The instantaneous phase can be introduced as the rotation angle of an image point around the origin on the plane (x, x_h) . In this case, it is necessary to ensure that the process $x(t)$ is stationary and integral (1) exists. For a dynamical chaos regime, the instantaneous phase $\Phi(t)$ can be introduced by considering the rotation angle of an image point on a plane of two properly selected coordinates, for example, on the plane of dynamical variables (x, y) . In this case, the equilibrium state of the dynamical system is set to be the origin of the radius-vector, around which the phase trajectory rotates. Then the instantaneous phase can be defined as follows:

$$\Phi(t) = \operatorname{arctg} \left(\frac{y(t)}{x(t)} \right) \pm \pi k, \quad k = 0, 1, 2, \dots \quad (4)$$

In [11] it is proposed to determine the instantaneous phase by using the plane of variables (\dot{x}, \dot{y}) which define the corresponding components of the phase velocity. Such a method may appear to be the most correct one as chaotic trajectories on this plane always rotate strictly around the origin of coordinates. The instantaneous phase can also be introduced using a sequence of time points t_k corresponding to the moments when the trajectory crosses a given secant plane. In this case, for arbitrary time moments, a stepwise, piecewise-linear or any other phase approximation is used [3, 4]. In a piecewise-linear approximation the instantaneous phase can be determined as follows:

$$\Phi(t) = \pi \frac{t - t_k}{t_{k+1} - t_k} \pm \pi k, \quad k = 0, 1, 2, \dots, \quad k = 0, 1, 2, \dots \quad (5)$$

Then, we can introduce the instantaneous frequency of oscillations

$$\omega = \frac{d\Phi(t)}{dt} \quad (6)$$

and the mean frequency

$$\bar{\omega} = \left\langle \frac{d\Phi(t)}{dt} \right\rangle = \lim_{t \rightarrow \infty} \frac{\Phi(t) - \Phi(t_0)}{t - t_0}, \quad (7)$$

where the brackets $\langle \dots \rangle$ denote time averaging.

Since the instantaneous phase introduced in various ways can have different values, statistical characteristics of the process $\Phi(t)$, for instance, mean frequency

values, may coincide. In this case, the particular method of the phase definition is of no importance. However, this coincidence cannot always be observed. The correctness and unambiguity of the phase description is of special importance if the chaos synchronization phenomenon is studied. Let us discuss in greater detail the synchronization of dynamical chaos. The generalized definition of phase synchronization proposed in [1, 2], which is applied to chaotic self-sustained oscillations, requires that the instantaneous phase difference be bounded for any t :

$$|n\Phi_1(t) - m\Phi_2(t)| < C, \quad (8)$$

where $\Phi_1(t)$ and $\Phi_2(t)$ are the instantaneous phases of interacting self-sustained oscillators (or a self-sustained oscillator and an external force), and n and m are the integers determining the synchronization order. Condition (8) written in terms of mean frequencies (7) is represented as follows:

$$n\bar{\omega}_1 - m\bar{\omega}_2 = 0, \quad (9)$$

where $\bar{\omega}_1$ and $\bar{\omega}_2$ are the mean frequencies of interacting chaotic self-sustained oscillators. The condition of the mean frequency locking $\bar{\omega}_1 = \bar{\omega}_2$ should correspond to the synchronization at the main tone ($n = m = 1$). However, the mean frequencies introduced in the above way do not always match typical maxima in the spectrum of aperiodic oscillations. In the latter case, a serious disagreement can be observed between the phase and frequency (spectral) methods used for description and diagnostics of the synchronization phenomenon.

2. Formulation of the Problem

The power spectra of chaotic oscillations in dynamical systems can be substantially different. The spectrum corresponding to the spiral chaos mode looks like the spectrum of noisy periodic oscillations with a well-pronounced peak at the basic frequency ω_0 [12–15]. The spectra of other types of chaos can be close to the one of white noise and do not contain pronounced maxima at any frequencies. The value of the mean frequency (7) depends on which instantaneous phase definition is used. If we remain within the framework of this formalism, then, while studying the synchronization phenomenon of complex self-sustained oscillations, we can obtain unreasonable results from a physical viewpoint. To avoid this in numerical and physical experiments, it is necessary to provide a detailed comparison of mean frequencies introduced in the frame of the phase dynamics approach with basic frequencies of the power spectrum. It seems impossible to reach this purpose for chaotic attractors with spectra being close to uniform ones. Therefore, it is worthwhile investigating the problem of interest with examples of spiral attractors whose spectrum exhibits a well-pronounced spectral line at the basic frequency, which can be measured experimentally. Thus, our objective is to compare two methods for studying the chaos synchronization, namely, the “phase dynamics” method and the spectral one, and to show their peculiarities. Recently, a considerable attention was paid to the phenomenon of forced and mutual synchronization in systems driven by noise (stochastic synchronization) [3, 4, 6, 8, 9, 16, 17]. The concept of stochastic synchronization can be based either on a partial mean switching frequency locking

or on the effect of instantaneous phase locking for sufficiently long time intervals (effective synchronization). Obviously, a correct definition of the instantaneous phase and the mean frequency will play a significant role in this case [16].

The objective of this paper is to establish a relationship between mean and basic frequencies of chaotic self-sustained oscillations and to clarify which methods of introducing the instantaneous phase and dynamic modes can bring them to coincidence. We intend to demonstrate when and for what reasons there is a difference between these frequencies, and how such a disagreement can affect the diagnostics of synchronization of chaotic self-sustained oscillations. We also consider the degree of agreement between different methods of introducing the instantaneous phase of stochastic oscillations in a bistable oscillator driven by an external force.

3. Frequency and Phase Characteristics of Chaotic Attractors in the Rössler System

We consider the classical example of the Rössler oscillator [18]. For certain parameter values, this well-known chaotic self-sustained oscillator can demonstrate both spiral and funnel chaos. The system equations read

$$\dot{x} = -y - z, \quad \dot{y} = x + \alpha y, \quad \dot{z} = \beta + z(x - \mu), \quad (10)$$

where $\alpha = \beta = 0.2$, and μ is the control parameter. We examine the behavior of the instantaneous phase of Eqs. (10), introduced according to definitions (3) and (4) for different variables. When expression (3) is used, the corresponding process is centralized. The mean frequency $\bar{\omega}$ is calculated using formula (7). In addition, we determine the basic frequency ω_0 of spectrum of chaotic oscillations. The calculation results have shown that the values of $\bar{\omega}$ corresponding to different methods of the instantaneous phase definition and the values of ω_0 do not always coincide. Moreover, such a disagreement was typically observed for the control parameter range $\mu > 6.2$. On the contrary, at $\mu \leq 6.2$ all the considered characteristics coincide within the limits of the calculation accuracy. The calculated dependencies $\bar{\omega}$ and ω_0 on the parameter μ are shown in Fig. 1. The dependence $\omega_0(\mu)$ was obtained for the $x(t)$ oscillations. However, as our calculations of the power spectra have demonstrated, the behavior of ω_0 is the same for all the dynamical variables.

As can be seen from Fig. 1, the values of $\bar{\omega}$ obtained for the x and y variables according to expression (3) (curves 1 and 2, respectively) differ significantly for $\mu > 6.2$ and do not equal to the values of ω_0 (curve 4) for the majority of the parameter μ values. In addition, Fig. 1 represents the calculation results for the mean frequency in the case when the instantaneous phase is defined as the rotation angle of a radius-vector in the (\dot{x}, \dot{y}) plane. The disagreement between the mean frequencies and the basic frequency is connected with the type of the chaotic attractor. Both frequencies typically coincide for the spiral chaos mode. As μ is increased, the spiral chaotic attractor is evolving and transforming into a funnel-type attractor [2, 3]. After a periodicity window in $7.9 \leq \mu \leq 8.1$, the funnel chaos mode is observed in the system. Although at some parameter μ values certain frequency dependencies can be rather close, on the whole, a significant divergence of the results is typical for this region. Let us consider in more detail the behavior of the instantaneous phase at $\mu = 6.5$. The chaotic attractor can still be referred to the spiral type but the

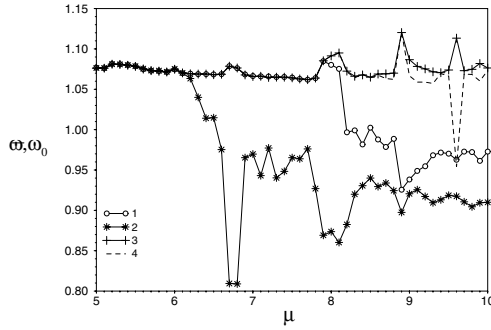


Fig. 1. Mean frequency $\bar{\omega}$ and basic frequency ω_0 of Eqs. (10) as functions of the parameter μ for $\alpha = \beta = 0.2$. Curves 1 and 2 reflect the behavior of mean frequencies of oscillations $x(t)$ and $y(t)$ when the instantaneous phase is introduced according to (3). Curve 3 corresponds to the mean frequency of the trajectory rotation around the origin in the (\dot{x}, \dot{y}) plane, and curve 4 illustrates the behavior of the basic frequency of the spectrum of oscillations $x(t)$.

attractor already demonstrates a complicated structure. All the mean frequencies calculated by various methods are equal with a high precision (up to 10^{-4}) to one another and correspond to the basic frequency ω_0 . This equality is violated only for the mean frequency of the $y(t)$ oscillations.

Now let us look at the power spectra. In the spiral chaos mode, the spectra of all dynamical variables (including $z(t)$ that is not appropriate for defining the instantaneous phase) are characterized by a well-pronounced maximum at the same basic frequency ω_0 [Fig. 2(a)]. The basic spectral line in the vicinity of ω_0 has the same shape for all observed variables, which is close to a Lorentzian [19], whereas a broadband pedestal of the spectrum depends on the choice of the observed one. The same is true for the funnel chaos. However, in this case, the Lorentzian with a maximum at ω_0 has a significant width [Fig. 2(b)].

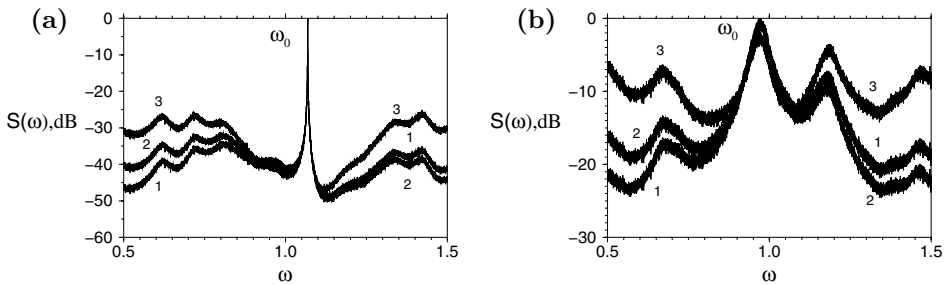


Fig. 2. Power spectra for Eq. (10): (a) for $\mu = 6.5$, and (b) for $\mu = 13$. Curves 1,2, and 3 were obtained for variables x, y, z , respectively.

Strictly speaking, frequencies of spectral maxima for a system with a continuous power spectrum are not invariant with respect to the choice of different variables. The coincidence of the basic frequencies in the case being studied does not follow from the fact that the spectral line width at the basic frequency is not large. Indeed, let us consider variables x and $\dot{x} = dx/dt$. Their power spectra are connected by

the relationship $S_{\dot{x}}(\omega) = \omega^2 S_x(\omega)$, and in the general case, the frequencies of the spectral maxima can be different. The spectral component being considered can be represented as follows [19]

$$S_x(\omega) = \frac{B}{B^2 + (\omega - \omega_0)^2}, \quad (11)$$

where B is the phase diffusion coefficient of chaotic oscillations. In this case the frequency of the spectrum $S_x(\omega)$ maximum is ω_0 , whereas the frequency of the $S_{\dot{x}}(\omega)$ maximum will be slightly different, i.e. $\omega_1 = \omega_0 + B^2/\omega_0$. However, this difference can be neglected and, consequently, we have $\omega_1 \approx \omega_0$.

As has been shown in [19], the coefficient B has a value of order 10^{-4} (in dimensionless units) for the spiral chaos mode, while it is of order 10^{-2} for the funnel chaos regime. All these estimations can change the frequency of the spectral maximum but only within the limits of the calculation accuracy. When the instantaneous phase is introduced in the plane (\dot{x}, \dot{y}) , the mean frequency values are most close to the basic frequency ω_0 in a wide range of the parameter μ values (see curve 3 in Fig. 1). However, in this case as well, the correspondence between the mean frequency and the basic frequency is not complete in the region of funnel chaos.

Thus, the basic frequency of oscillations in the spiral chaos mode appears to be an invariant characteristic with respect to the selection of the observed variable. For the successful determination of the instantaneous phase, the mean frequency $\bar{\omega}$ should coincide with ω_0 . If this coincidence is absent, the phase behavior does not reflect peculiarities of chaotic self-sustained oscillations. We will illustrate this statement by considering the forced harmonic synchronization of the Rössler oscillator.

4. Phase and Spectral Representations of the Chaos Synchronization

Now, we consider whether the spectral criterion of synchronization (multiplicity of basic spectrum frequencies) coincides with the phase criterion (8). As an example, we study the forced synchronization of the chaotic Rössler oscillator by a harmonic signal. The corresponding system of equations can be written as follows

$$\dot{x} = -y - z + C \sin(\omega_{ex}t), \quad \dot{y} = x + \alpha y, \quad \dot{z} = \beta + z(x - \mu), \quad (12)$$

where ω_{ex} and C are the frequency and amplitude of the external signal, respectively. We consider the synchronization at the main tone, i.e. for the following frequency relationship $\omega_0 = \omega_{ex} = 1$. Criterion (8) is equal to mean frequency $\bar{\omega}$ locking. If the instantaneous phase $\Phi(t)$ is introduced correctly, the following equality should be met:

$$\theta = \frac{\bar{\omega}}{\omega_{ex}} = \frac{\omega_0}{\omega_{ex}} = 1, \quad (13)$$

where θ is the rotation number. When different phase definitions lead to the same value of frequency $\bar{\omega}$ being equal to the basic frequency, there is no difference what way is used to determine the rotation number. However, if we choose an “unsuccessful” phase definition method, we can fail to register the synchronization effect. As an example, Fig. 3 illustrates the behavior of the phase difference between oscillations and the external signal, $\Delta\Phi(t) = \Phi(t) - \omega_{ex}t$, in the synchronization region of the spiral chaos at $\mu = 6.5$.

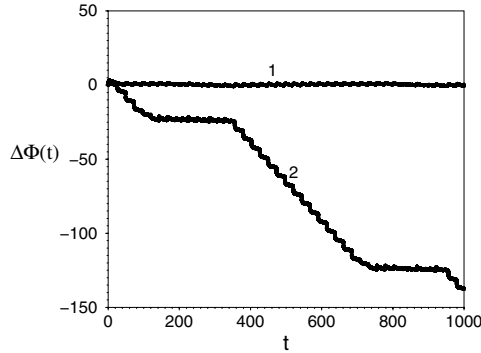


Fig. 3. Phase difference $\Delta\Phi(t)$ in Eq. (12) in the synchronization region for $\mu = 6.5$, $C = 0.05$, and $\omega_{ex} = 1.068$. Curves 1 and 2 correspond to instantaneous phases $\Phi(t)$ obtained for variables x and y , respectively.

Curve 1 corresponds to the instantaneous phase $\Phi(t)$ calculated for variable x and indicates the phase locking. Curve 2 is found for variable y . In the autonomous mode with the chosen value of μ , the mean frequency of oscillations $y(t)$ do not coincide with the basic frequency and the phase behavior is nonmonotonic. Hence, curve 2 in Fig. 3 does not reflect the phase locking effect. Figure 4 illustrates dependencies of the rotation number on frequency ω_{ex} obtained for the spiral chaos mode at $\mu = 6.5$. Rotation number θ is calculated according to the instantaneous phase definition (3) for variables x (curve 1) and y (curve 3) as well as using the basic frequency ω_0 (curve 2). We can observe a good coincidence of curves 1 and 2 which display the same locking region $\theta(\omega_{ex}) = 1$. However, when dynamic variable y is used, the mean frequency and, hence, the rotation number do not reflect the synchronization phenomenon (curve 3).

It is obvious that the synchronization phenomenon itself is independent of the coordinate chosen for observation. A characteristic should exist which allows one to diagnose this phenomenon for any selected variable. This characteristic can be the basic spectrum frequency ω_0 . Indeed, as ω_{ex} is varied, basic spectral maxima in spectra of variables $x(t)$ and $y(t)$ (in contrast to $\bar{\omega}$) evolve similarly. For a given amplitude of the external signal, the basic frequency ω_0 is locked at the external frequency ω_{ex} .

Thus, the absence of mean frequency $\bar{\omega}$ locking diagnosed numerically does not mean the absence of the synchronization yet. It can appear that an unsuccessful variable was chosen for calculating the instantaneous phase and the corresponding frequency. At the same time, in the region of the developed funnel chaos the synchronization does not actually exist. This fact can be demonstrated by analyzing power spectra of Eq. (12) in the funnel chaos mode. When the external frequency changes, the spectral line corresponding to ω_{ex} shifts against the background of a fixed spectral maximum at the basic frequency. It should be noted that the mutual synchronization of self-sustained oscillators of the funnel chaos seems to be possible (at least, in the effective sense) [11, 20].

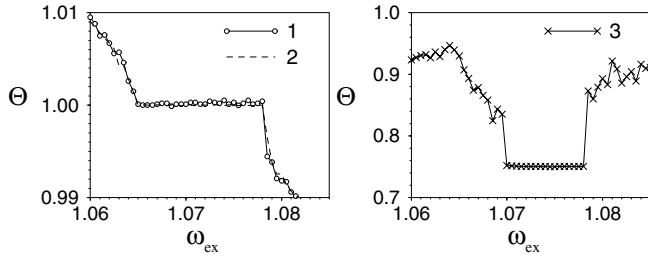


Fig. 4. Rotation number as a function of ω_{ex} for $\mu = 6.5$ and $C = 0.05$ for different ways of defining θ : curve 1 characterizes the ratio of frequencies $\bar{\omega} : \omega_{ex}$ for variable x ; curve 2 the ratio $\omega_0 : \omega_{ex}$; and curve 3 the ratio $\bar{\omega} : \omega_{ex}$ for variable y .

5. Instantaneous Phase and Mean Frequency of a Bistable Stochastic Oscillator

Now let us discuss the problem regarding to the agreement between different methods of defining the instantaneous phase and mean frequency in the case of stochastic oscillations. As an example we consider the Kramers oscillator:

$$\dot{x} = x - x^3 + \sqrt{2D}\xi(t). \tag{14}$$

It describes the overdamped motion of a Brownian particle in a double-well potential $U(x) = -x^2/2 + x^4/4$. The particle is driven by a random force in the form of uncorrelated Gaussian process $\sqrt{2D}\xi(t)$ with zero mean. The noise causes the particle to switch between the wells by overcoming a potential barrier ΔU . The mean switching frequency (the Kramers rate) depends on the barrier height and constant D denoting the noise intensity. In the case of a sufficiently high potential barrier and weak noise, the Kramers rate is defined by the following law:

$$r_K = \eta \exp\left(-\frac{\Delta U}{D}\right). \tag{15}$$

The prefactor η is given by the curvature of the potential wells. For model (14) we have $\Delta U = 1/4$ and $\eta = 1/(\sqrt{2\pi})$. We study numerically the random process $x(t)$ and its approximation by the telegraph signal $x_T(t)$, which corresponds to the switchings between levels ± 1 at the moments of falling within the region $|x| \geq 0.5$. The statistics of switchings is still retained when passing to the process $x_T(t)$ but the information on the system intrawell dynamics is lost. The instantaneous phase of stochastic switchings can be calculated by using three different methods: (i) according to definition (3) for the process $x(t)$ immediately; (ii) according to expression (5), and (iii) from expression (3) for the telegraph signal $x_T(t)$ corresponding to the process $x(t)$. The obtained results have shown that the phase behavior can differ for various methods of the phase definition. As can be seen from Fig. 5(a), the first way of defining the instantaneous phase leads to a more rapid growth of the phase in time compared to methods (ii) and (iii).

We use the calculation results for the instantaneous phase defined by methods (i)–(iii) to find the corresponding mean frequencies $\omega_{1,2,3}$ from definition (7). The

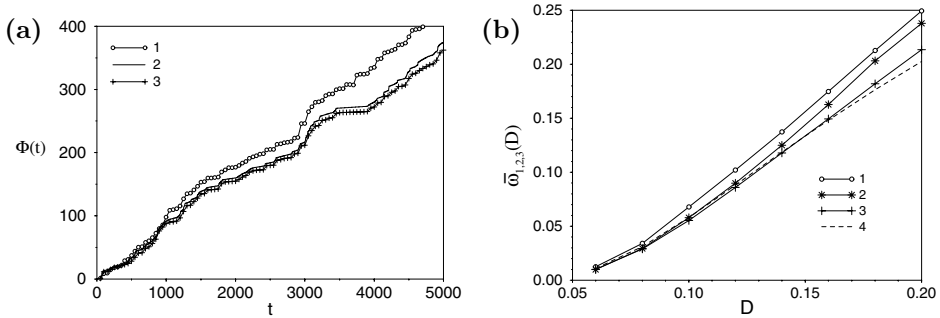


Fig. 5. (a) Temporal dependence of the phase of stochastic oscillations in Eq. (14) at $D = 0.1$ for different methods of the phase definition: according to formula (3) for the process $x(t)$ immediately (curve 1), from expression (5) (curve 2), and according to (3) for the telegraph signal $x_T(t)$ (curve 3), and (b) Mean frequencies $\omega_{1,2,3}$ as functions of the noise intensity D (curves 1, 2 and 3, respectively). The dashed line (curve 4) shows the theoretical dependence of the Kramers rate on D .

mean switching frequency determined immediately as $\lim_{T \rightarrow \infty} N/T$, where N is the number of switchings in one direction during observation time T , must coincide with the frequency ω_2 . Indeed, this coincidence with a sufficiently good precision can be observed in numerical experiments. The mean frequencies as functions of the noise intensity D are shown in Fig. 5(b) together with theoretical dependence $\omega_K = \pi r_K$, where the Kramers rate r_K is defined by formula (15). The frequency values are quite close for weak noise and differ essentially when the noise intensity increases. The difference of mean frequency values, introduced by different ways, has also been shown in [16] for a harmonic oscillator driven by noise.

If the noise intensity is not large (when formula (15) is valid), a good agreement can be observed between the theoretical dependence of the frequency on D (curve 4) and the numerical results obtained for frequencies $\omega_{2,3}$ (curves 2 and 3). At the same time, the first method of phase definition leads to a higher value of the mean frequency (curve 1). As the noise intensity D increases, all the calculated mean frequencies begin to differ substantially. When two stochastic oscillators interact, the synchronization of random switchings can be observed [3, 4, 6, 8]. In this case, the effect of instantaneous phase locking exists for large time intervals. However, since these time intervals are finite, we can speak only about the so-called effective synchronization. The latter effect can be usually registered by analyzing either the effective phase diffusion coefficient that must have values close to 0, or the mean frequency ratio that should be about 1. The width of the effective synchronization region is determined at a given diffusion level or at a certain allowed frequency difference. However, as follows from the obtained results, the width of the synchronization region may be different for various methods of the phase definition. Thus, a method of introducing the instantaneous phase can substantially affect results of the studies of the stochastic synchronization effect.

6. Conclusions

In the present paper we have compared two methods for diagnosing the synchronization phenomenon, e.g. the “phase dynamics” method and the spectral one. It has

been shown that both methods can lead, as a rule, to the same results in the case of chaotic oscillations with a high degree of phase coherence (spiral chaos). However, we have established that the method of phase dynamics can depend essentially on the way of the instantaneous phase definition and on the choice of the observable. This peculiarity can especially manifest itself when passing to more complicated oscillatory regimes, i.e. the funnel chaos mode. Thus, using only one method for studying the effect of chaos synchronization is insufficient and it is desirable to utilize different methods and then compare obtained results.

Different methods of introducing the instantaneous phase also give significantly different results in the case of stochastic switchings in bistable oscillators. Thus, the way of the instantaneous phase definition can influence the estimation of the width of the effective synchronization region of stochastic oscillators.

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