Multistability of partially synchronous regimes in a system of three coupled logistic maps

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Abstract— **We research the bifurcations of partially synchronous periodic orbits which lead to the multistability formation in a system of three coupled logistic maps with symmetrical diffusive coupling. We demonstrate that the partially synchronous regimes appeare as results of saddle-node bifurcations. The mechanisms of stability loss of partial synchronization of chaos are discussed.**

I. Introduction

Nonlinear oscillatory systems often demonstrate the phenomenon of multistability, when several attractors coexist in the phase space at the same parameters values. Such systems attract a great interest of researchers because of their promising possibilities for the aim of control of oscillatory dynamics. The typical example of a system with developed multistability is a small ensemble of diffusively coupled period-doubling oscillators. As it was shown in a number of works [1]-[3], two coupled period-doubling oscillators demonstrate a great variety of multistable regular and chaotic regimes. In our previous investigations, we have found that the mechanism of the multistability formation in such systems is connected with the phenomenon of loss of complete chaotic synchronization [4]. The process of the synchronism breaking at parameters change is appeared to be induced by the same bifurcations of the principal periodic solutions that lead to the formation of new stable regimes in the systems phase space. Is this situation typical for the systems with higher dimension? How the process of multistability formation interacts with the synchronization phenomenon in ensembles of more than two oscillators? In attempt to answer these questions we consider an ensemble with one more oscillator: a ring of three coupled period-doubling maps. It demonstrates phenomena of both complete and partial chaotic synchronization. The last case denotes that only two oscillators in the ring are synchronized while the behavior of the third one remains unsynchronous. Interdependence between complete and partial synchronization in systems of three oscillators with differnet types of coupling, bifurcations that lead to the synchronism breaking, as well as the phenomena of bubbling of the attractors and the riddled basins which accompany them, have been considered in a number of works [5]-[9]. In our investigations we concentrate on the role of this bifurcations in the process of multistability formation. We describe typical chains of bifurcations which lead to formation of hierarchy of partially synchronous regimes.

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II. System Under Consideration

We consider a system of three coupled identical logistic maps in the form:

$$\begin{aligned} x_{n+1} &= f(x_n) + \frac{\gamma}{2}\Big(f(y_n) + f(z_n) - 2f(x_n)\Big) \\ y_{n+1} &= f(y_n) + \frac{\gamma}{2}\Big(f(x_n) + f(z_n) - 2f(y_n)\Big) \\ z_{n+1} &= f(z_n) + \frac{\gamma}{2}\Big(f(x_n) + f(y_n) - 2f(z_n)\Big) \end{aligned} \tag{1}$$

where $f(x) = \lambda - x^2$. A single map demonstrates transition to chaos through a cascade of period-doubling bifurcations with λ increasing. The coupled maps system has the following types of synchronous behavior: complete synchronization, when trajectory belongs to the symmetric subspace ($x = y = z$) and three kinds of partial synchronization, when trajectory belongs to one of the subspaces ($x = y,\ x \neq z$), ($x = z,\ x \neq y$) or ($y = z,\ x \neq y$). Because the all kinds of partial synchronization posses identical properties, further we will consider only one of them: ($x = y$, $x \neq z$).

Any synchronous motions can be observed in experiments only if they are stable to perturbations transversal to the corresponding symmetric subspaces. These stability properties are usually analyzed by transversal Lyapunov exponents $\Lambda_\perp^{c/p}$ (Λ^c – for the complete synchronization, Λ^p – for the partial synchronization), which for the system under study have the following form:

$$\Lambda_\perp^{c/p} = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \ln f'(x_n) + \ln\left(1 - \frac{3\gamma}{2}\right) \tag{2}$$

where the values of x_n belongs to the chaotic trajectory located inside the corresponding subspaces of symmetry: in the diagonal $x = y = z$ for the complete synchronization, or in the two-dimensional plane $x = y$ for the partial synchronization. Let us considered the eq. (2) separately for the cases of complete and partial synchronization.

- In the case of the complete synchronization the dynamics of x_n satisfies the equation of a single map $x_{n+1} = f(x_n)$ and, hence, the first term in the (2) represents the tangent Lyapunov exponent: $\Lambda_\tau^c = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \ln f'(x_n)$. If the coupling parameter is positive and not too large ($0 < \gamma < \frac{2}{3}$), the value of transversal Lyapunov exponent is smaller then the tangent one. Consequently, all synchronous regular regimes are transversally stable, while the synchronous chaotic regimes are stable only at sufficiently strong coupling.

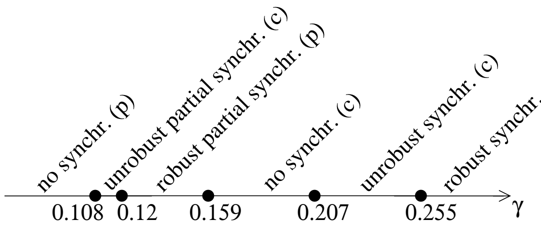

Fig. 1. Order of regimes emerging with decrease of coupling (”c” – chaotic behaviour, ”p” – periodic behaviour). $\lambda = 1.67$.

As the synchronous dynamics is one-dimensional, and, accordingly, the subspace, that is complementary to the symmetrical one, is two-dimensional, there are two transversal Lyapunov exponents. They are equal to each other and both are determined by the equation (2). Consequently, all bifurcations that take place in the transversal direction are singular: the bifurcational condition is satisfied for two characteristic exponents simultaneously.

- On the contrary, in the case of partial synchronization dynamics of a system is two-dimensional, it satisfies the equations:

$$x_{n+1} = f(x_n) + \frac{\gamma}{2}\Big(f(z_n) - f(x_n)\Big) \qquad (3)$$
$$z_{n+1} = f(z_n) + \gamma\Big(f(x_n) - f(z_n)\Big),$$

Since the complementary subspace is one-dimensional there is only one transversal Lyapunov exponent. Bifurcations leading to the transversal instability of partially synchronous regimes are not singular.

III. From Full to Partial Synchronization

Let us consider the interdependence between complete and partial synchronization of chaos in the system (1). We chose the value of the parameter $\lambda = 1.67$, that relates to one-band chaotic attractor in the single map. The analysis of the equation (2) demonstrates that both partially and completely synchronous regimes are stable at sufficiently strong coupling ($\gamma > 0.207$). With the decrease of coupling, when Λ^c becomes positive ($\gamma = 0.207$) the transition from complete to unsynchronous regime is observed. Next, with further γ decrease, the transversal Lyapunov exponent for the regime of partial synchronization Λ^p becomes negative and the trajectory moves to one of the planes of partial synchronization. With further decrease of coupling the regime of partial synchronization loses its stability, and the system transits to unsynchronous state. The order of the regimes with the coupling decrease is shown in Fig. 1. From the regime of strong complete chaotic synchronization to the regime of strong partial chaotic synchronization the behavior of the system goes through the following stages: strong regime of complete chaotic synchronization ($\gamma > 0.255$) $\rightarrow$ weak regime of complete synchronization of chaos ($\gamma > 0.207$) $\rightarrow$ unsynchronous chaotic regime ($\gamma > 0.159$) $\rightarrow$ strong regime of partial synchronization of regular motions ($\gamma > 0.12$) $\rightarrow$ strong partial synchronization of chaos ($0.108 < \gamma < 0.12$). With further decrease of coupling the oscillations become unsynchronous. When we speak about ”strong” and ”weak” synchronization we mean that strong synchronization is robust to small noise or parameters mismatch, while weak synchronization is unrobust to them: in regimes of weak synchronization an addition of small noise results in ”bubbling of attractor” phenomenon, when a trajectory occasionally leave the symmetric subspace. That phenomenon is shown on the example of a weak full synchronization regime on Fig. 2. Now, we consider the bifurcational mechanism of loss of complete synchronization and formation of partially synchronous regimes in the system. Similarly to the case of two maps [10], we research the bifurcations of the principal saddle periodic orbits, i.e. the orbits, on the base of which the considered one-band synchronous chaotic attractor has been formed. We begin with the synchronous orbit of period one $1C^0$ (Fig. 3). At $\lambda = 1.67$ and sufficiently strong coupling $\gamma \geq 0.29$ the orbit $1C^0$ is saddle: it is unstable in the tangent to the symmetrical subspace direction and stable transversally to it. At $\gamma = 0.29$ this orbit undergoes a period-doubling bifurcation, at which two multipliers become equal to -1 simultaneously. The correspondent eigenvectors are directed transversally to symmetric subspace. They form the basis of invariant subspace ($x + y + z = 0$). As a result of this bifurcation the orbit $1C^0$ loses its transversal stability and becomes a repeller. In its vicinity three saddle orbits of doubled period $2C^{1,i}$ ($i = 1, 2, 3$) appeared, which are symmetric to each other at cyclic coordinate change (Fig. 3). These orbits do not belong to any of the subspaces of symmetry, and remain unstable for any parameters values. The one-directional unstable manifolds of the orbits $2C^{1,i}$ direct along the planes of partial symmetry to the points of synchronous

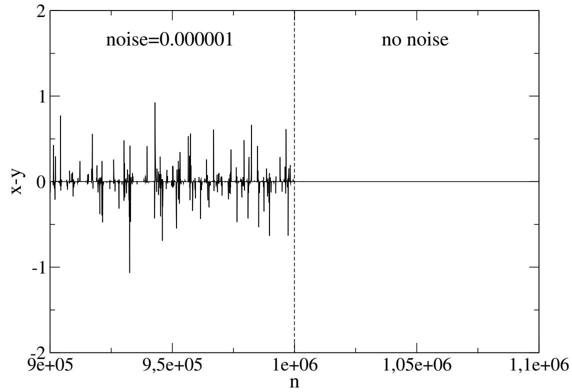

Fig. 2. Influence of a noise on the weak regime of full synchronization. $\lambda = 1.67$, $\gamma = 0.22$

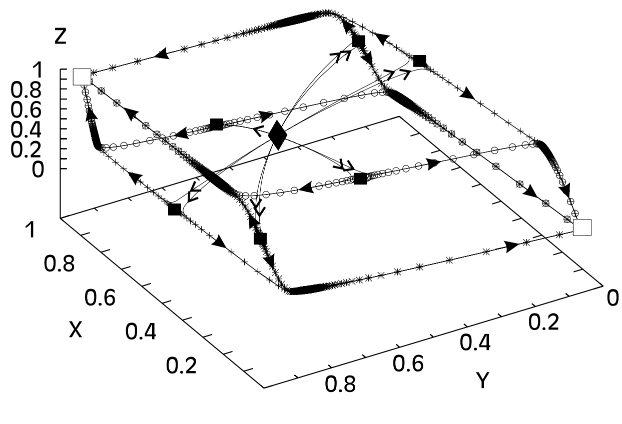

Fig. 3. Location of periodic orbits taking part in the destruction of the complete synchronization of chaos: $1C^0$ (♦), $2C^0$ (□), $2C^{1,i}$ (■).

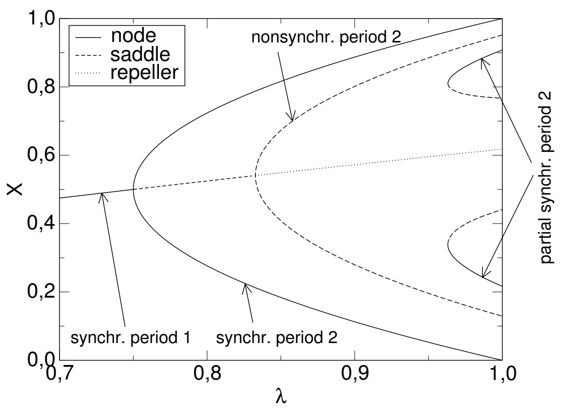

Fig. 4. Formation of strong partial synchronization regime and its basins of attraction. $\gamma = 0.05$

orbit $2C^0$ (Fig. 3). The appearance of the repeller in the symmetrical subspace initiates the weak synchronization, because it form the region of a local transversal instability. From that region the phase points leave the symmetrical subspace along the transversal unstable manifold of the repeller $1C^0$ to one of the saddle periodic orbits $2C^{1,i}$, and then return to the vicinity of the symmetrical subspace because of global attraction to it (the value of the transversal Lyapunov exponent $\Lambda_\perp^c$ remains negative). With further decrease of coupling the orbits of longer periods 2^nC^0, $n = 1, 2, 3, ...$ undergo similar bifurcations: $2^nC^0 \rightarrow 4^nC^1$. In the result, in the system with noise the bubbling phenomena becomes stronger. Finally, for $\gamma = 0.21$ the transversal Lyapunov exponent becomes positive. The blowout bifurcation takes place, as a result of which the regime of complete synchronization of chaotic motion becomes unstable. There is no more attraction either to the diagonal $x = y = z$, nor to any of the planes of partial synchronization. The system demonstrates unsynchronous regime.

With further decrease of γ a saddle-node bifurcation takes place in every subspace of partial symmetry on the contour which is formed by unstable manifolds of one of the orbits $2C^{1,i}$ (Fig. 3). As a result a stable partially synchronous orbits of period two $2C_1^i$ appears (Fig. 4). The appeared orbit originates a family of regular and chaotic regimes of strong partial synchronization, that is formed through the period-doubling route. The similar bifurcations take place on the manifolds of unstable periodic orbits $4^nC^{1,i}$ appeared from the synchronous orbits of the main family 2^nC^0. In its turn, the each stable orbit $4^nC^{1,i}$ originates the own family of partially synchronous regimes. Thus, the resulted structure of the parameters space becomes very complex, with overlapping regions of stability of the partially synchronous regimes related to different families.

IV. Multistubility Formation

To investigate the coexistence of partially synchronous regimes it is convenient to use the "truncated" equations (3). Since its phase space is two-dimensional, there are two multipliers which determine the stability of regimes of partial synchronization inside the plane of partial symmetry. We consider bifurcations that lead to appearance of partially synchronous regimes on the plane of the both systems parameters λ and γ. We have found three different mechanisms of formation of regimes of partial synchronization in the system (3). One of them was discussed above. It follows from the saddle-node bifurcation on the line l_1^2 (Fig. 5), on which the partially synchronous orbit of period two ($2C_1$) emerges. On the line l_{11}^4 the first multiplier of the orbit $2C_1$ becomes equal to -1 and in its vicinity a stable partially synchronous orbit of doubled period $4C_{11}$ emerges. The region of stability of the orbit $4C_{11}$ on the plane of parameters is formed by line l_{11}^4, line l_{111}^8 on which the period-doubling bifurcation takes place, and line l_0^4, where the saddle-node bifurcation takes place. On the line l_1^{2s} (Fig. 5) the second multiplier of the saddle partially synchronous orbit $2C_1$ becomes equal to -1, and in its vicinity saddle partially synchronous orbit of period four ($4C_{11}^s$) emerges. With further increase of parameter λ on the line l_{12}^4 as a result of a saddle-node bifurcation the stable orbit $4C_{12}$ emerges. The orbit $4C_{11}^s$ forms the one of the borders of the basins of attraction of the orbit $4C_{12}$. With further increase of the parameter λ both partially synchronous orbits $4C_{11}$ and $4C_{12}$ undergoes period-doubling bifurcations. On the line l_{111}^8 (Fig. 5) the first multiplier of the orbit $4C_{11}$ becomes equal to -1, the period-doubling bifurcation takes place, and in its vicinity the partially synchronous orbit $8C_{111}$ emerges. On the line l_{11}^{4s} the second multiplier of the orbit $4C_{11}$ becomes equal to -1, and the orbit $8C_{111}^s$ emerges. Further parameters change leads to the saddle-node bifurcation on the line l_{112}^8, in the result of which an orbit $8C_{112}$ emerges. The same bifurcations take place with the orbit $4C_{12}$ on the lines accordingly l_{121}^8, l_{12}^4s and l_{121}^8.

Thus, on the base of the partially synchronous orbit of period two ($2C_1$) there appeared two orbits of period four, four orbits of period eight and so on. Similar bifurcations are observed for the orbits which appeared on the base of

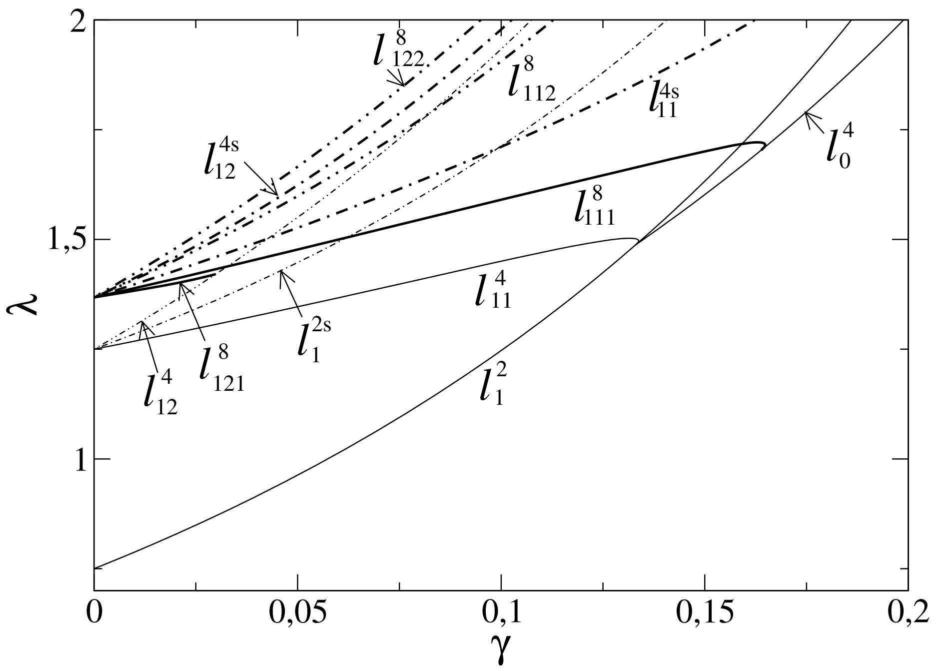

Fig. 5. Regions of stability of partially synchronous regimes based on the partially synchronous orbit of period two.

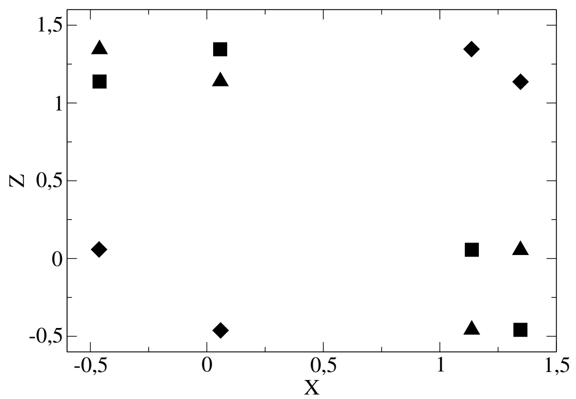

Fig. 6. The partially synchronous orbits of period 4: $4C_1$ (◆), $4C_{11}$ (▲), $4C_{12}$(■). $\lambda = 1.35$, $\gamma = 0.001$

the orbits of period four ($4C_1$). This orbit undergoes the period doubling bifurcation as a result of which the partially synchronous orbit of period 8 ($8C_{11}$) appears. The second orbit of period 8 ($8C_{12}$) emerges as a result of a saddle-node bifurcation, its basins of attraction are formed by the saddle periodic orbit ($4C_1$) and the saddle periodic orbit which emerged as a result of saddle-node bifurcation with the cycle ($8C_{12}$). The same mechanisms are observed for the orbits of a longer period. On the Fig. 6 the partially synchronous attractors of period four are shown. The attractor $4C_1$ on the Fig. 6 is appeared as a result of the saddle-node bifurcation on the unstable manifolds of unsynchronous saddle periodic orbits of period two, the attractor $4C_{11}$ emerged as a result of period-doubling bifurcation of the partially synchronous orbit of period two ($2C_1$), and the attractor ($4C_{12}$) on the Fig. 6 appeared as a result of a saddle-node bifurcation, unlike the orbit $4C_1$, its basins of attraction are formed by the saddles inside the plane of symmetry. That attractors coexist for the same values of parameters, and are separated in the phase space. The all above-mentioned phenomena take place also for the orbits of longer periods. As a result all the partially synchronous stable regimes form a complicated picture of multistability. On the other hand, from the system (3) we can observe the dynamics of the original system (1) only inside the chosen plane of partial synchronization. However, besides the bifurcations inside this plane the system (1) loses its stability in transversal direction. When these bifurcations take place with periodic orbits, in their vicinity the unsynchronous orbits of the doubled period emerge. Next, the pitchfork bifurcation takes place and that unsynchronous orbits become stable. The basins of attraction of these unsynchronous orbits locate in the vicinity of the partially synchronous chaotic attractor, and as a result the riddled basins phenomenon is observed. Besides the riddling of basins of attraction one can observe the bubbling phenomenon. That two phenomena follows the stability loss of partial chaotic synchronization, after which the phase point moves away from the subspace of symmetry.

V. Conclusions

We have researched the phenomenon of partial synchronization in a system of three coupled logistic maps. The phenomena of appearance of partially synchronous regimes, bifurcations inside and outside the synchronization plane and the phenomenon of partial synchronization loss have been considered. The strong partial synchronization regime emerges as a result of the saddle-node bifurcation which takes place on the unstable manifolds of unsynchronous orbits, which emerged as a result of period-doubling bifurcation outside the symmetric subspace of the synchronous orbits. The period of partially synchronous orbits emerged that way depends on the controlling parameters values. Next, through the cascade of bifurcations the transition to a strong partial synchronization of chaos is observed. Besides the period-doubling bifurcations on the transition to chaos saddle-node bifurcations take place inside the plane of synchronization and new partially synchronous periodic orbits emerge. As a result the multistability of different partially synchronous regimes is observed inside the synchronization plane. Besides the bifurcations inside the subspace of partial symmetry the orbits can lose their stability transversally to the plane of partial symmetry. As a result unsynchronous saddle periodic orbits emerge. That orbits with further variation of parameters as a result of a pitchfork bifurcation become stable. That phenomena result in multistability formation outside the synchronization plane, and induce the bubbling and the riddled basins phenomena on the loss of stability of partial synchronization of chaos. As a result of partial synchronization loss the unsynchronous motions emerge.

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