Fluctuation and Noise Letters
Vol. 3, No. 2 (2003) L213–L221

SPECTRAL AND CORRELATION ANALYSIS OF SPIRAL CHAOS

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Received 17 January 2003
Revised 6 March 2003
Accepted 25 March 2003

We study numerically the behavior of the autocorrelation function (ACF) and the power spectrum of spiral attractors without and in the presence of noise. It is shown that the ACF decays exponentially and has two different time scales. The rate of the ACF decrease is defined by the amplitude fluctuations on small time intervals, i.e., when $\tau < \tau_{\rm cor}$, and by the effective diffusion coefficient of the instantantaneous phase on large time intervals. It is also demonstrated that the ACF in the Poincare map also decreases according to the exponential law $\exp(-\lambda^+ k)$, where λ^+ is the positive Lyapunov exponent. The obtained results are compared with the theory of fluctuations for the Van der Pol oscillator.

Keywords: Autocorrelation function; power spectrum; effective phase diffusion coefficient.

1. Introduction

The problem of decay of autocorrelation functions (ACF) in continuous dynamical systems with dimension $N \geq 3$ is one of the fundamental and still unresolved tasks of the theory of chaos. In a common case this problem has not been still resolved theoretically due to the presence of certain significant difficulties. For two-dimensional discrete systems that satisfy the Smale axiom-A it has been only proven that the ACF can be bounded from above by an exponentially decreasing function [1–5]. In certain cases the ACF decays exponentially with the exponent that is defined by the inverse of the Kolmogorov entropy $h = \lambda^+$, where λ^+ is the positive Lyapunov exponent [4]. In a common case, such statements are not fulfilled for even hyperbolic maps. Regularities of the ACF decay in differential systems with chaotic attractors of both hyperbolic and nonhyperbolic types are even more complicated from a viewpoint of their theoretical description. As was shown in [6–8], the rate of the ACF decay in differential systems depends essentially on the structure of an attractor and on the influence of noise. Moreover, the positive Lyapunov exponent does not define the regularities of decay of autocorrelations [7,8]. In the present Letter we substantiate numerically that for typical nonhyperbolic attractors of the spiral type in R^3, the autocorrelations decay exponentially. With this, two time scales can be distinguished, i.e., $\tau \leq \tau_{\text{cor}}$ and $\tau > \tau_{\text{cor}}$. In the first case the exponential decay is defined by fluctuations of the instantaneous amplitude and in the second case it depends on the effective diffusion coefficient B_{eff}.

The power spectrum of spiral or phase-coherent chaos exhibits a pronounced peak at the basic (average) frequency and, consequently, the envelope of the ACF decreases relatively slow [9–12]. Spiral attractors can be observed in such well-known systems as the Rössler oscillator [13], the Anishchenko–Astakhov generator [11], or the Chua circuit [14]. The self-sustained oscillations in these systems can remind the dynamics of noisy periodic oscillators of a Van der Pol oscillator type [15–19]. The main objective of our Letter is to substantiate quantitatively that chaotic attractors of the spiral type possess the properties of a noisy limit cycle, although, these attractors are realized in fully deterministic systems, i.e., without external fluctuations.

2. Classical Theory of the Van der Pol Oscillator

Spectral and correlation properties of quasiharmonic self-sustained oscillations in the presence of noise were studied in the framework of the classical theory of fluctuations in the Van der Pol oscillator [20–23]. If the oscillations $x(t)$ are represented in the form of $x(t) = A(t)\cos(\omega_0 t + \phi(t))$, then after averaging we obtain the well-known reduced stochastic equations for the Van der Pol oscillator [20]:

$$\begin{aligned} \dot{A} &= \gamma A\left(1 - \frac{A^2}{A_0^2}\right) + \frac{\omega_0^2 D}{A} + \omega_0\sqrt{2D}\nu(t)\,, \\ \dot{\phi} &= \frac{\omega_0\sqrt{2D}}{A}\eta(t)\,. \end{aligned} \tag{1}$$

Here γ is the parameter of excitation, ω_0 is the frequency, and A_0 is the unperturbed amplitude. In the regime of generation the mean amplitude $\langle A(t)\rangle$ can be considered to be equal to A_0. $\nu(t)$ and $\eta(t)$ are independent sources of δ-correlated noise which have a Gaussian distribution with zero mean. The parameter D governs the noise intensity. If the amplitude fluctuations $\tilde{A}(t) = A(t) - A_0$ are small, the phase dynamics can be approximated by the following equation:

$$\dot{\phi} = 2B\eta(t)\,, \qquad B = \frac{\omega_0\sqrt{2D}}{2A_0}\,. \tag{2}$$

Equation (2) describes a Wiener process with the diffusion coefficient B. It is important to note that in the approximation considered, the amplitude $A(t)$ and phase $\phi(t)$ are treated as statistically independent random quantities. In doing so the ACF for a stationary random process $\psi_x(\tau) = \langle x(t)x(t+\tau)\rangle - \langle x(t)\rangle^2$ can be written as follows [20–23]:

$$\psi_x(\tau) = \frac{1}{2} K_A(\tau) \exp\left(-B|\tau|\right) \cos \omega_0 \tau \,, \tag{3}$$

where $K_A(\tau) = \langle A(t)A(t+\tau)\rangle = \psi_A(\tau) + A_0^2$ is the covariance function of the instantaneous amplitude. The autocorrelation function of the amplitude $\psi_A(\tau)$ can be expressed as $\psi_A(\tau) = \langle (A(t) - A_0)^2 \rangle \exp\left(-2\gamma|\tau|\right)$ and, consequently, for the power spectrum density $S_x(\omega)$ we have

$$S_x(\omega) = \frac{BA_0^2}{B^2 + (\omega - \omega_0)^2} + \frac{(2\gamma + B)\langle (A(t) - A_0)^2 \rangle}{(2\gamma + B)^2 + (\omega - \omega_0)^2} . \tag{4}$$

As can be seen from Eqs. (3) and (4), the influence of noise leads to a decrease of the ACF $\psi_x(\tau)$, to a finite spectral line width at the frequency of auto-generation (the first term in Eq. (4)) and to the occurrence of a noisy background (the second term in Eq. (4)) [20–23]. An increase of the spectral line and the reduction of the ACF basically depend on the phase diffusion coefficient B, while the noisy background is defined by the ACF of the instantaneous amplitude $\psi_A(\tau)$, which is connected with the parameter of excitation $\gamma \gg B$. Now we use this concept to the characterization of spiral chaos.

3. Spectral and Correlation Analysis of Spiral Chaos

We start with the Rössler system:

$$\dot{x} = -y - z + \sqrt{2D}\xi(t) \,, \qquad \dot{y} = x + \alpha y \,, \qquad \dot{z} = \beta + z(x - \mu) \,, \tag{5}$$

where $\xi(t)$ is the normalized Gaussian source of δ-correlated noise with zero mean and D is the noise intensity. We fix $\alpha = \beta = 0.2$ and $\mu = 6.5$. Let us introduce the change of variables

$$x(t) = A(t) \cos \Phi(t) \,, \qquad y(t) = A(t) \sin \Phi(t) \,, \tag{6}$$

which determines the amplitude $A(t)$ and the full phase $\Phi(t)$ of the chaotic oscillations. Substituting (6) Eqs. (5) can be re-written as follows:

$$\begin{aligned} \dot{A} &= \frac{1}{2}\alpha A - \frac{1}{2}\alpha A \cos 2\Phi - z \cos \Phi + \sqrt{2D}\xi(t) \cos \Phi \,, \\ \dot{\Phi} &= 1 + \frac{1}{2}\alpha \sin 2\Phi + \frac{1}{A} z \sin \Phi - \frac{\sqrt{2D}}{A}\xi(t) \sin \Phi \,, \\ \dot{z} &= \beta + z(A \cos \Phi - \mu) \,. \end{aligned} \tag{7}$$

In our numerical calculations we use both systems (5) and (7).

In [8] it has been recently shown that for spiral chaos in the Rössler system the variance $\sigma_\phi^2(t)$ of the instantaneous phase grows linearly in time both without noise ($D = 0$) and when $D \neq 0$. The variance of the total phase is equal to the variance of its non-regular component $\phi(t) = \Phi(t) - \bar{\omega}t$, where $\bar{\omega} = \langle \dot{\Phi}(t) \rangle$ is the mean frequency of the chaotic oscillations. This linear dependence of variance $\sigma_\phi^2(t)$ on time allows us to introduce the effective phase diffusion coefficient

$$B_{\mathrm{eff}} = \frac{1}{2}\frac{\mathrm{d}\sigma_\phi^2(t)}{\mathrm{d}t}\,. \tag{8}$$

In our numerical simulation of Eqs. (5) we calculate the normalized autocorrelation function of chaotic oscillations $\Psi_x(\tau) = \psi_x(\tau)/\psi_x(0)$. Using Eqs. (7) we compute the covariance function of the amplitude $K_A(\tau)$ and the effective phase diffusion coefficient B_{eff}. We use the time-averaging procedure for calculating $\Psi_x(\tau)$ and $K_A(\tau)$. The coefficient B_{eff} is computed by averaging over an ensemble of realizations [8]. Figure 1 shows the calculation results for $\Psi_x(\tau)$ in system (5). The ACF decays almost exponentially both without noise (Fig. 1(a)) and in the presence of noise (Fig. 1(b)). Additionally, as seen from Fig. 1(c), for $\tau < 20$ there is an interval on which the correlations decrease much faster.

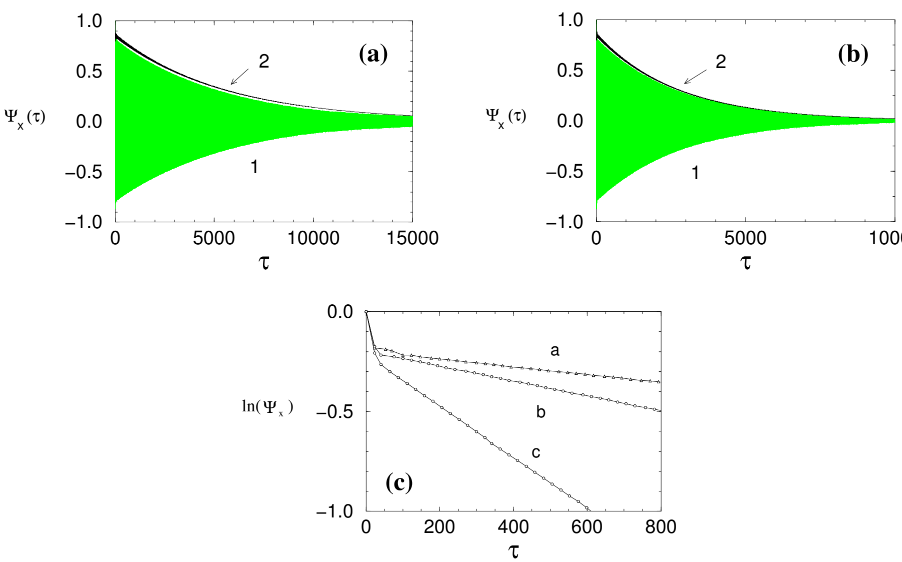

Fig. 1. Calculation results for the Rössler system (5) with $\alpha = \beta = 0.2$ and $\mu = 6.5$. Normalized ACF $\Psi_x(\tau)$ (curve 1) and the approximation of its envelope (9) (curve 2) for (a) $D = 0$, $B_{\mathrm{eff}} = 0.00019$ and (b) $D = 0.001$, $B_{\mathrm{eff}} = 0.00032$. (c) The envelopes of $\Psi_x(\tau)$ in a logarithmic scale for $D = 0$ (curve a), $D = 0.001$ (curve b), and $D = 0.01$ (curve c).

Using Eq. (3) we can approximate the envelope of the calculated ACF $\Psi_x(\tau)$. To do this, we substitute the numerically computed characteristics $K_A(\tau)$ and $B = B_{\mathrm{eff}}$ into an expression for the normalized envelope $\Gamma(\tau)$:

$$\Gamma(\tau) = \frac{K_A(\tau)}{K_A(0)}\exp\left(-B|\tau|\right). \tag{9}$$

The calculation results for $\Gamma(\tau)$ are shown in Fig. 1(a,b) by black dots (curves 2). It is seen that the behavior of the envelope of $\Psi_x(\tau)$ is described well by Eq. (9). Note that taking into account the multiplier $K_A(\tau)/K_A(0)$ enables us to obtain a good approximation both for large time intervals ($\tau \geq 20$) and on the interval $0 < \tau < 20$. This means that the amplitude fluctuations play a significant role on short time intervals, while the slow process of correlation splitting is mainly determined by the phase diffusion. Thus, we observe a surprisingly good agreement between the numerical results for the spiral chaos and the theoretical data for the quasiharmonic self-sustained generator. It is important to note that the data presented in Fig. 1(a) are obtained in the regime of deterministic chaos, i.e., without external fluctuations. Additionally, our numerical results have demonstrated that the mutual correlation function between the amplitude and phase fluctuations is significantly larger than zero.

We have shown that for $\tau > \tau_{\mathrm{cor}}$ the envelope of ACF can be approximated by the exponential law $\exp(-B_{\mathrm{eff}}|\tau|)$. Then according to the Wiener-Khinchin theorem, the spectral peak at the average frequency $\bar{\omega}$ must have a Lorenzian shape and be defined by the first term of Eq. (4) for $B = B_{\mathrm{eff}}$ and $\omega_0 = \bar{\omega}$ [a]. The numerical findings are presented in Fig. 2. The basic spectral peak is approximated by using (4) and this fits quite well with the calculation results for the power spectrum obtained both without and in the presence of noise. The ACF $\Psi_x(\tau)$, its envelope and the power spectrum have also been estimated for different noise intensities $0 < D < 10^{-2}$ and for the range of parameter μ values corresponding to the regime of spiral attractor. These findings have demonstrated a close similarity to the results presented in Figs. 1 and 2.

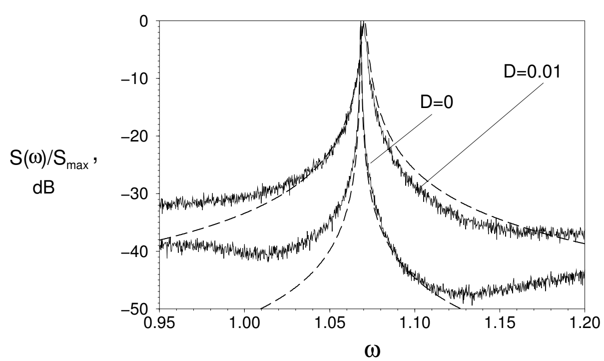

Fig. 2. Power spectra near the basic frequency (solid lines) and their theoretical approximations (4) (dashed lines) for the Rössler attractor (5) for $D = 0$ and $D = 0.01$.

As has been recently established in [7, 8] and follows from Fig. 1, the decrease of the ACF in chaotic continuous-time systems in R^3 is not determined by the positive Lyapunov exponent. Nevertheless, the question on the interrelation between mixing and the λ^+ remains to be answered. Our investigations have shown that the splitting of correlations can be connected with the λ^+. This interrelation can be clearly observed for the dynamics of a Poincaré section. The numerical results (Fig. 3) indicate that the normalized ACF $\Psi_X(k) = \psi_X(k)/\psi_X(0)$ for the Poincaré

[a]This fact requires special numerical calculations due to the presence of two time scales in the ACF and the non-zero correlation between the amplitude and phase fluctuations in Eq. (7).

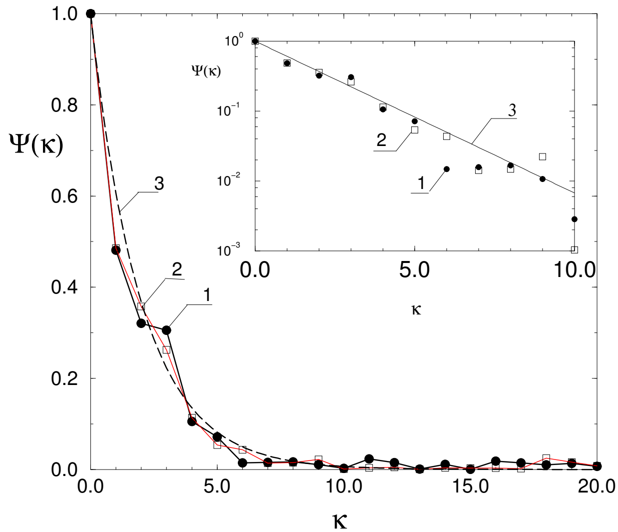

Fig. 3. The ACF of a chaotic realization in the Poincaré section ($y = 0$) of the Rössler system for $D = 0$ (curve 1), $D = 0.001$ (curve 2), and the approximation (curve 3). The same results in a logarithmic scale are given in the inlet.

section $y = 0$ in the system (5) decays according to the exponential approximation $\exp(-\lambda^+ k)$, where $\lambda^+ = 0.51$.

As follows from Fig. 3, the agreement between the numerical results and the exponential approximation is not so good as in Fig. 1. This can be mainly explained by a finite accuracy in the construction of the Poincaré section and an insufficiently long sequence of points in the map. It is well-known that without noise the Poincaré section can be computed precisely by using the Henon algorithm [24]. However, this algorithm cannot be used in the presence of noise [25]. We also assume that there can be a deeper reason. The exponential decay of the ACF is proven for maps which satisfy the Smale axiom. The spiral attractor in the system (5) and, consequently, the attractor in the corresponding Poincaré section are typically nonhyperbolic.

4. Spectral and Correlation Characteristics of the GIN

Our findings for the approximation of the ACF and the shape of the basic spectral peak are completely confirmed by our investigations of spiral attractors in different dynamical systems. We exemplify this for the Anishchenko–Astakhov generator (GIN) for which there is a good agreement between the results of numerical and full-scale experiments [11]. The GIN model is governed by the following equations:

$$\dot{x} = mx + y - xz + rx^3 + \sqrt{2D}\xi(t)\,, \qquad \dot{y} = -x\,, \qquad \dot{z} = -gz + gf(x)\,. \tag{10}$$

Here $f(x) = x^2$ for $x > 0$, and $f(x) = 0$ for $x \le 0$. $\xi(t)$ is the noise source with the same characteristics as in the system (5). The control parameters are fixed as $m = 1.35$ and $g = 0.21$. We have studied both the system (10) and equations obtained after applying the change of variables (6). Our numerical calculations of the ACF, its envelope and the power spectrum for the GIN have shown a good

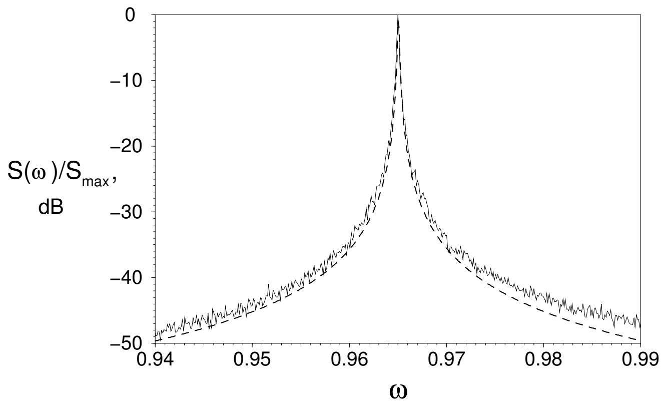

Fig. 4. Power spectrum near the basic frequency (solid line) and its theoretical approximation (4) (dashed line) for the spiral attractor in system (10) for $D = 10^{-5}$, $B_{\mathrm{eff}} = 0.00148$.

qualitative agreement with the results obtained for the Rössler system and presented in Figs. 1 and 2. For example, Fig. 4 illustrates the power spectrum and the approximation results for the basic spectral peak for the chosen noise intensity. It demonstrates a better coincidence of the theoretical (4) and numerical findings as compared with Fig. 2. We note that for the funnel (non-coherent) type of chaotic attractors (for example, for $m = 10$ in the Rössler system (5)) the results being similar to Figs. 2 and 4 cannot be obtained, although the ACF also decreases exponentially on large time intervals.

5. Conclusion

In conclusion, we have shown in our numerical simulation that the spiral chaos retains to a great extent the spectral and correlation properties of a noisy Van der Pol oscillator. With this, the rate of correlation splitting in the regime of spiral chaos is determined by the amplitude fluctuations and the instantaneous phase diffusion. In turn the effective phase diffusion coefficient in a noise-free system is defined by its chaotic dynamics and is not related directly to the positive Lyapunov exponent. Basing on the presented results we can state that the model of a stochastic process in the form of a harmonic oscillation with random amplitude and phase can describe sufficiently well the spectral and correlation properties of chaotic attractors of the spiral type both in a purely dynamical case and in the presence of noise (see Fig. 1). Noisy oscillations of the Van der Pol oscillator can also be considered as a particular case of such a process. It is important to note that for a chaotic attractor, generated by the spiral chaos in the Poincaré map, the rate of the ACF decay is described by the exponential law $\exp(-\lambda^+ k)$, where λ^+ is the positive Lyapunov exponent of the attractor in the Poincaré map.

The obtained results are highly important from both fundamental and applied viewpoints. In particular, they can be applied to solving the problems of predictability and reconstruction of attractors, which are very important in various fields, for instance, meteorology, finances, ecology, economics, etc.

Acknowledgements

We are grateful to Prof. A. Pikovsky for valuable discussions and G. Okrokvertskhov for his help in carrying out numerical calculations. This work was partially supported by Award No. REC-006 of the U.S. Civilian Research and Development Foundation. V.S.A. acknowledges support from the RFBR (grant No. 00-02-17512). T.E.V. acknowledges support from INTAS (grant No. 01-867) and G.I.S. acknowledges support from INTAS (grant No. YSF 2002-3).

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