

V. Anishchenko, A. Neiman, T. Vadivasova,  
V. Astakhov, and L. Schimansky-Geier

# Dynamics of Chaotic and Stochastic Systems

Tutorial and Modern Development

*Second edition: improved and enlarged*

May 3, 2007

Springer-Verlag

Berlin Heidelberg New York

London Paris Tokyo

Hong Kong Barcelona

Budapest

To our teachers:

Werner Ebeling, Yuri L. Klimontovich  
and Frank Moss



## Preface to the Second Edition

We present an improved and enlarged version of our book V.S. Anishchenko et al. *Nonlinear Dynamics of Chaotic and Stochastic Systems* published by Springer in 2002. In its basis the new edition of the book corresponds to its first version. When preparing the present edition we made some clarifications in several sections as well as corrected the noticed misprints in formulas. Besides, three new sections were added to Chapter 2. They are "Statistical Properties of Dynamical Chaos", "Effects of Synchronization in Extended Self-Sustained Oscillatory Systems", and "Synchronization in Living Systems". The indicated sections naturally enroll in the material of the second chapter and reflect the most interesting results obtained by the authors after publication of the first edition.

We hope that the new edition of the book will be of great interest for a wide auditorium of readers who are already specialists or only beginning researchers in the fields of nonlinear oscillation and wave theory, dynamical chaos, synchronization and stochastic process theory.

Saratov, Berlin, and St. Louis  
April 2006

*V.S. Anishchenko*  
*A.B. Neiman*  
*T.E. Vadiavasova*  
*V.V. Astakhov*  
*L. Schimansky-Geier*



---

## Preface to the First Edition

This book is devoted to the classical background and to contemporary results on nonlinear dynamics of deterministic and stochastic systems. Considerable attention is given to the effects of noise on various regimes of dynamical systems with noise-induced order.

On the one hand, there exists a rich literature of excellent books on nonlinear dynamics and chaos; on the other hand, there are many marvelous monographs and textbooks on the statistical physics of far-from-equilibrium and stochastic processes. This book is an attempt to combine the approach of nonlinear dynamics based on the deterministic evolution equations with the approach of statistical physics based on stochastic or kinetic equations. One of our main aims is to show the important role of noise in the organization and properties of dynamic regimes of nonlinear dissipative systems.

We cover a limited region in the interesting and still expanding field of nonlinear dynamics. Nowadays the variety of topics with regard to deterministic and stochastic dynamic systems is extremely large. Two main criteria were followed in writing the book and to give a reasonable and closed presentation: (i) the dynamic model should be minimal, that is, most transparent in the physical and mathematical sense, and (ii) the model should be the simplest which nevertheless clearly demonstrates most important features of the phenomenon under consideration.

The book consists of three chapters. The first chapter serves as a brief introduction, giving the fundamental background of the theory of nonlinear deterministic and stochastic systems and a classical theory of the synchronization of periodic oscillations. All basic definitions and notions necessary for studying the subsequent chapters without referring to special literature are presented.

The second chapter is devoted to deterministic chaos. We discuss various scenarios of chaos onset including the problem of the destruction of two- and three-frequency quasiperiodic motion. Different aspects of synchronization and chaos control as well as the methods of reconstruction of attractors and dynamic systems from experimental time series are also discussed.

The third chapter is concerned with stochastic systems whose dynamics essentially depend on the influence of noise. Several nonlinear phenomena are discussed: stochastic resonance in dynamic systems subjected to harmonic

and complex signals and noise, stochastic synchronization and stochastic ratchets, which are the noise-induced ordered and directed transport of Brownian particles moving in bistable and periodic potentials. Special attention is given to the role of noise in excitable dynamics.

The book is directed to a large circle of possible readers in the natural sciences. The first chapter will be helpful for undergraduate and graduate students in physics, chemistry, biology and economics as well as for lecturers of these fields interested in modern problems of nonlinear dynamics. Specialist of nonlinear dynamics may use this part as an extended dictionary. The second and the third chapters of the book addresses to specialists in the field of mathematical modelling of the complex dynamics of nonlinear systems in the presence of noise.

We tried to write this book in such a manner that each of the three chapters can be understood in most parts independently of the others. Particularly, each chapter has its own list of references. This choice is based on the desire to be helpful to the reader. Undoubtedly, the lists of references are incomplete. We indicate the most important contributions on each particular subject studied and give references to papers whose results were directly used in writing the book.

This book is a result of the long-term collaboration of the Nonlinear Dynamics Laboratory at Saratov State University, the group of Applied Stochastic Processes of Humboldt University at Berlin, and the Center for Neurodynamics at the University of Missouri at St. Louis. We want to express our deep gratitude to W. Ebeling, Yu.L. Klimontovich and F. Moss for their support, scientific exchange and constant interest. We acknowledge fruitful discussions with C. van den Broeck, P. Hänggi, J. Kurths, A. Longtin, A. Pikovski, and Yu.M. Romanovski. The book has benefited a lot from our coauthors of the original literature. We wish to thank A. Balanov, R. Bartussek, V. Bucholtz, I. Dikstein, J.A. Freund, J. García-Ojalvo, M. Hasler, N. Janson, T. Kapitaniak, I. Khovanov, M. Kostur, P. S. Landa, B. Lindner, P. McClintock, E. Mosekilde, A. Pavlov, T. Pöschel, D. Postnov, P. Reimann, R. Rozenfeld, P. Ruzsyczynsky, A. Shabunin, B. Shulgin, U. Siewert, A. Silchenko, O. Sosnovtseva, A. Zaikin and C. Zülicke for regular and fruitful discussions, criticism and valuable remarks which give us deeper insight into the problems we study.

We are especially grateful to Miss Galina Strelkova for her great work in preparing the manuscript and for translating several parts of this book into English.

V. Anishchenko, T. Vadivasova, and V. Astakhov acknowledge support from the U.S. Civilian and Development Foundation (CRDF) under grant No. REC-006; V.S. Anishchenko acknowledges support from the Alexander von Humboldt Foundation. A. Neiman was supported by the Fetzer Institute and by the office of Naval Research, Physics Division. L. Schimansky-Geier

acknowledges support from the Deutsche Forschungsgemeinschaft (Sfb 555 and GK 268).

Saratov, Berlin and St. Louis  
September 2001

*V.S. Anishchenko*  
*A.B. Neiman*  
*T.E. Vadiavasova*  
*V.V. Astakhov*  
*L. Schimansky-Geier*





# Contents

<b>1. Tutorial</b> .....	1
1.1 Dynamical Systems .....	1
1.1.1 Introduction .....	1
1.1.2 The Dynamical System and Its Mathematical Model ..	1
1.1.3 Stability – Linear Approach .....	8
1.1.4 Bifurcations of Dynamical Systems, Catastrophes .....	17
1.1.5 Attractors of Dynamical Systems. Deterministic Chaos	28
1.1.6 Summary .....	36
1.2 Fluctuations in Dynamical Systems .....	36
1.2.1 Introduction .....	36
1.2.2 Basic Concepts of Stochastic Dynamics .....	38
1.2.3 Noise in Dynamical Systems .....	46
1.2.4 The Fokker–Planck Equation .....	55
1.2.5 Stochastic Oscillators .....	61
1.2.6 The Escape Problem .....	68
1.2.7 Summary .....	78
1.3 Synchronization of Periodic Systems .....	79
1.3.1 Introduction .....	79
1.3.2 Resonance in Periodically Driven Linear Dissipative Oscillators .....	80
1.3.3 Synchronization of the Van der Pol Oscillator. Classical Theory .....	82
1.3.4 Synchronization in the Presence of Noise. Effective Synchronization .....	91
1.3.5 Phase Description .....	95
1.3.6 Summary .....	100
<b>References</b> .....	100
<b>2. Dynamical Chaos</b> .....	107
2.1 Routes to Chaos .....	107
2.1.1 Introduction .....	107
2.1.2 Period-Doubling Cascade Route. Feigenbaum Univer- sality .....	108

2.1.3	Crisis and Intermittency . . . . .	116
2.1.4	Route to Chaos via Two-Dimensional Torus Destruction	119
2.1.5	Route to Chaos via a Three-Dimensional Torus. Chaos on $T^3$ . Chaotic Nonstrange Attractors . . . . .	128
2.1.6	Route to Chaos via Ergodic Torus Destruction. Strange Nonchaotic Attractors . . . . .	131
2.1.7	Summary . . . . .	137
2.2	Statistical Properties of Dynamical Chaos . . . . .	137
2.2.1	Introduction . . . . .	137
2.2.2	Diagnosis of Hyperbolicity in Chaotic Systems . . . . .	139
2.2.3	Chaos in the Presence of Noise . . . . .	141
2.2.4	Relaxation to a Stationary Probability Distribution for Chaotic Attractors in the Presence of Noise . . . . .	142
2.2.5	Spectral-Correlation Analysis of Dynamical Chaos . . . . .	148
2.2.6	Phase Diffusion in an Active Inhomogeneous Medium Described by the Ginzburg–Landau Equation . . . . .	153
2.2.7	The Autocorrelation Function and Power Spectrum of Spiral Chaos in Physical Experiments . . . . .	158
2.2.8	Summary . . . . .	161
2.3	Synchronization of Chaos . . . . .	162
2.3.1	Introduction . . . . .	162
2.3.2	Phase–Frequency Synchronization of Chaos. The Classical Approach . . . . .	163
2.3.3	Complete and Partial Synchronization of Chaos . . . . .	169
2.3.4	Phase Multistability in the Region of Chaos Synchroni- zation . . . . .	174
2.3.5	Bifurcation Mechanisms of Partial and Complete Chaos Synchronization Loss . . . . .	178
2.3.6	Summary . . . . .	183
2.4	Effects of Synchronization in Extended Self-Sustained Oscil- latory Systems . . . . .	183
2.4.1	Introduction . . . . .	183
2.4.2	Cluster Synchronization in an Inhomogeneous Chain of Quasiharmonic Oscillators . . . . .	185
2.4.3	Effect of Noise on Cluster Synchronization in a Chain of Quasiharmonic Oscillators . . . . .	188
2.4.4	Cluster Synchronization in an Inhomogeneous Self- Sustained Oscillatory Medium . . . . .	193
2.4.5	Cluster Synchronization in Interacting Inhomogeneous Media . . . . .	198
2.4.6	Forced Synchronization of a Chain of Chaotic Self- Sustained Oscillators . . . . .	200
2.4.7	Synchronization and Multistability in a Ring of An- harmonic Oscillators . . . . .	204

2.4.8	Synchronization and Multistability in a Ring of Oscillators with Period Doubling . . . . .	213
2.4.9	Summary . . . . .	220
2.5	Synchronization in Living Systems . . . . .	221
2.5.1	Introduction . . . . .	221
2.5.2	Stochastic synchronization of electroreceptors in the paddlefish . . . . .	222
2.5.3	Synchronization of Cardiorhythm . . . . .	225
2.5.4	Summary . . . . .	228
2.6	Controlling Chaos . . . . .	230
2.6.1	Introduction . . . . .	230
2.6.2	Controlled Anti-Phase Synchronization of Chaos in Coupled Cubic Maps . . . . .	232
2.6.3	Control and Synchronization of Chaos in a System of Mutually Coupled Oscillators . . . . .	239
2.6.4	Controlled Chaos Synchronization by Means of Periodic Parametric Perturbations . . . . .	244
2.6.5	Stabilization of Spatio-Homogeneous Motions by Parametric Perturbations . . . . .	248
2.6.6	Controlling Chaos in Coupled Map Lattices . . . . .	251
2.6.7	Summary . . . . .	260
2.7	Reconstruction of Dynamical Systems . . . . .	261
2.7.1	Introduction . . . . .	261
2.7.2	Reconstruction of Attractors from Time Series . . . . .	263
2.7.3	Global Reconstruction of DS . . . . .	272
2.7.4	Reconstruction from Biological Data . . . . .	278
2.7.5	Global Reconstruction in Application to Confidential Communication . . . . .	284
2.7.6	Summary . . . . .	289
	<b>References</b> . . . . .	290
<b>3.</b>	<b>Stochastic Dynamics</b> . . . . .	303
3.1	Stochastic Resonance . . . . .	303
3.1.1	Introduction . . . . .	303
3.1.2	Stochastic resonance: Physical background . . . . .	305
3.1.3	Characteristics of SR . . . . .	307
3.1.4	Response on a weak signal. Theoretical approaches . . . . .	309
3.1.5	Array enhanced stochastic resonance . . . . .	316
3.1.6	Doubly Stochastic Resonance SR in systems with noise induced phase transition . . . . .	327
3.1.7	Stochastic resonance for signals with a complex spectrum . . . . .	332
3.1.8	Stochastic resonance in chaotic systems with coexisting attractors . . . . .	340
3.1.9	Analog simulation . . . . .	345

3.1.10	Summary . . . . .	346
3.2	Synchronization of Stochastic Systems . . . . .	348
3.2.1	Introduction . . . . .	348
3.2.2	Synchronization and Stochastic Resonance . . . . .	349
3.2.3	Forced stochastic synchronization of the Schmitt trigger	356
3.2.4	Mutual stochastic synchronization of coupled bistable systems . . . . .	360
3.2.5	Forced and mutual synchronization of switchings in chaotic systems . . . . .	363
3.2.6	Stochastic Synchronization of Ensembles of Stochastic Resonators . . . . .	368
3.2.7	Stochastic synchronization as noise-enhanced order . . .	373
3.2.8	Summary . . . . .	376
3.3	The Beneficial Role of Noise in Excitable Systems . . . . .	377
3.3.1	Coherence Resonance Near Bifurcations of Periodic Solutions of a Dynamical System . . . . .	377
3.3.2	Coherence Resonance in Excitable Dynamics . . . . .	379
3.3.3	Noise-enhanced synchronization of coupled excitable systems . . . . .	390
3.3.4	Summary . . . . .	394
3.4	Noise Induced Transport . . . . .	395
3.4.1	Introduction . . . . .	395
3.4.2	Flashing and Rocking Stochastic Ratchets . . . . .	397
3.4.3	The Adiabatic Approach . . . . .	400
3.4.4	The Overdamped Correlation Ratchet . . . . .	402
3.4.5	Particle Separation by Ratchets, Driven by Colored Noise . . . . .	404
3.4.6	Two-dimensional Ratchets . . . . .	410
3.4.7	Discrete Ratchets . . . . .	414
3.4.8	Sawtooth-like media . . . . .	420
3.4.9	Making Spatial Structures by Ratchets . . . . .	424
3.4.10	Summary . . . . .	429
	<b>References . . . . .</b>	<b>429</b>